J Stat Phys (2010) 139: 565–597
DOI 10.1007/s10955-010-9954-3

# A Bijection Theorem for Domino Tilings with Diagonal Impurities

**Fumihiko Nakano · Taizo Sadahiro**

Received: 17 July 2009 / Accepted: 21 March 2010 / Published online: 1 April 2010


**Abstract** We consider the dimer problem on a planar non-bipartite graph $G$, where there are two types of dimers one of which we regard as impurities. Computer simulations reveal a reminiscence of the Cheerios effect, that is, impurities are attracted to the boundary, which is the motivation to study this particular graph. Our main theorem is a variant of the Temperley bijection: a bijection between the set of dimer coverings and the set of spanning forests with certain conditions. We further discuss some implications of this theorem: (1) the local move connectedness yielding an ergodic Markov chain on the set of all possible dimer coverings, and (2) a rough bound for the number of dimer coverings and that for the probability of finding an impurity at a given edge, which is an extension of a result in (Nakano and Sadahiro in arXiv:0901.4824).

**Keywords** Dimer model · Impurity · Temperley bijection

## 1 Introduction

Let $G = (V(G), E(G))$ be a graph. We say that a subset $M$ of $E(G)$ is a dimer covering of $G$ if it satisfies the following condition: "for any $x \in V(G)$ we can uniquely find $e \in M$ with $x \in e$". Each element $e \in M$ is called a dimer in $M$. Many results have been obtained on the dimer covering since the pioneering works [8, 9, 18], and it is still a topic of active research and rapid development (e.g., [10] and references therein). However, dimer problems on non-bipartite graphs are not so studied; one of the reason may be that it is hard to find an

F. Nakano (✉)
Department of Mathematics and Information Science, Kochi University, 2-5-1 Akebonomachi, Kochi, 780-8520, Japan
e-mail: nakano@math.kochi-u.ac.jp

T. Sadahiro
Faculty of Administration, Prefectural University of Kumamoto, Tsukide 3-1-100, Kumamoto, 862-8502, Japan
e-mail: sadahiro@pu-kumamoto.ac.jp

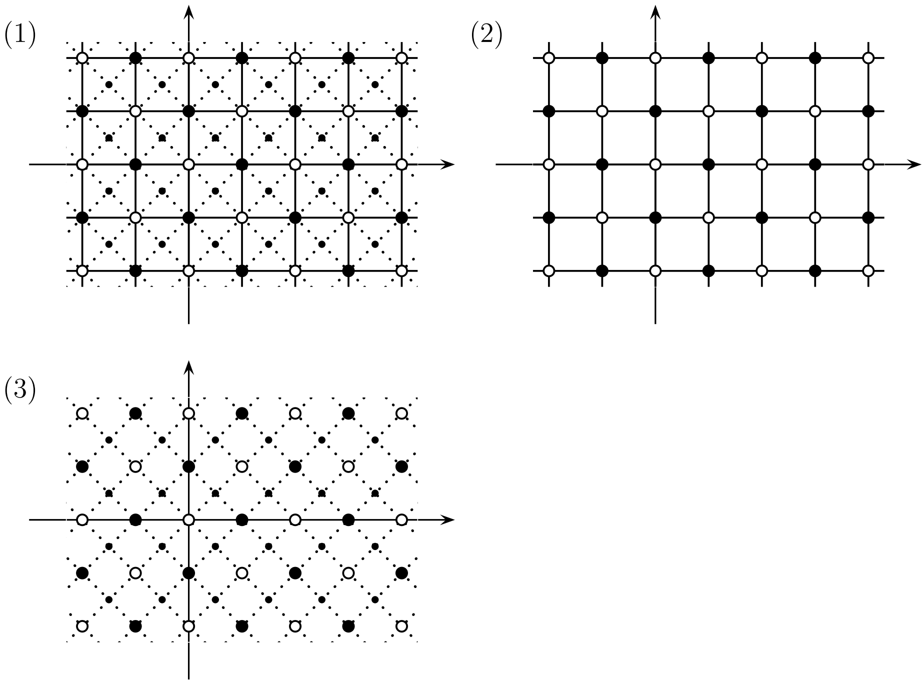



**Fig. 1** (**1**) The graph $\mathcal{G}$, (**2**) square lattice $\mathcal{G}_0$, (**3**) square lattice $\mathcal{G}'_0$

appropriate notion of the height function [19], and thus we may need an alternative to study the global structure.

In this paper, we consider two kinds ($G^{(m,n)}$, $G^{(k)}$ to be introduced below) of graphs both of which are finite subgraphs of $\mathcal{G} := R(\mathbf{Z}^2)$. $R(\mathbf{Z}^2)$ is the radial graph of $\mathbf{Z}^2$ defined as follows (Fig. 1(1)).

$$\mathcal{V} := V(\mathcal{G}) = \mathbf{Z}^2 \cup \left(\mathbf{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right)\right),$$

$$\mathcal{E} := E(\mathcal{G}) = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{V}, |\mathbf{x} - \mathbf{y}| = 1/\sqrt{2},\ 1\}.$$

The set $\mathcal{V}$ of vertices of $\mathcal{G}$ consists of three disjoint subsets: $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ where

$$\mathcal{V}_1 := \{(x, y) \in \mathbf{Z}^2 \mid x - y \in 2\mathbf{Z}\},$$

$$\mathcal{V}_2 := \{(x, y) \in \mathbf{Z}^2 \mid x - y \in 2\mathbf{Z} + 1\},$$

$$\mathcal{V}_3 := \mathbf{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right).$$

In Fig. 1(1), white circles are vertices in $\mathcal{V}_1$, and black circles (resp. black dots) are those in $\mathcal{V}_2$ (resp. $\mathcal{V}_3$). $\mathcal{V}_1 \cup \mathcal{V}_2 =: V(\mathcal{G}_0)$ is the vertex set of a square (and hence bipartite) lattice $\mathcal{G}_0$ (Fig. 1(2)), and $\mathcal{V}_3$ is the set of points located at the center of faces of $\mathcal{G}_0$. Bipartiteness of $\mathcal{G}_0$ means $\mathcal{V}_1$ and $\mathcal{V}_2$ satisfy the following condition: for $x, y \in V(\mathcal{G}_0)$, $(x, y) \in E(\mathcal{G}_0)$ implies $x \in \mathcal{V}_1$, $y \in \mathcal{V}_2$ or $x \in \mathcal{V}_2$, $y \in \mathcal{V}_1$. Unless stated otherwise, we do not consider orientation on

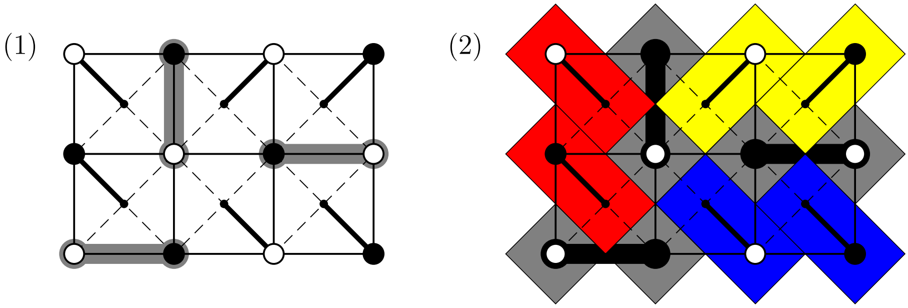



**Fig. 2** (**1**) An example of finite subgraph of $\mathcal{G}$ and its dimer covering. Impurities are those on vertical or horizontal edges. (**2**) The dimer covering of $G$ is equivalent to the tiling of the corresponding region by dominoes and impurities

edges and identify $e = (x, y)$ with $\overline{e} = (y, x)$. The edge set $\mathcal{E} = E(\mathcal{G})$ of $\mathcal{G}$ is divided into two disjoint subsets $\mathcal{E}_1$ and $\mathcal{E}_2$.

$$\begin{aligned}
\mathcal{E} &= \mathcal{E}_1 \cup \mathcal{E}_2, \\
\mathcal{E}_1 &:= \{(x, y) \in E(\mathcal{G}) \mid x \in \mathcal{V}_1 \cup \mathcal{V}_2,\ y \in \mathcal{V}_3\}, \\
\mathcal{E}_2 &:= \{(x, y) \in E(\mathcal{G}) \mid x, y \in \mathcal{V}_1 \cup \mathcal{V}_2\}.
\end{aligned}$$

In Fig. 1(1), dotted lines are edges in $\mathcal{E}_1$, while solid ones are those in $\mathcal{E}_2$. $\mathcal{G}$ can also be regarded as a graph obtained by adding diagonal edges (those in $\mathcal{E}_2$) in alternate directions to a square lattice $\mathcal{G}_0'$ (Fig. 1(3)), whose vertex and edge sets are given by $V(\mathcal{G}_0') = V(\mathcal{G})$, $E(\mathcal{G}_0') = \mathcal{E}_1$.

Let $G(\subset \mathcal{G})$ be a finite subgraph of $\mathcal{G}$ and let $M$ be a dimer covering of $G$. A dimer $e \in M \cap \mathcal{E}_1$ is also a dimer in a finite subgraph $G \cap \mathcal{G}_0'$ of the square lattice $\mathcal{G}_0'$. In this respect, it may be natural to call the dimers $e \in M \cap \mathcal{E}_2$ impurities. For instance in Fig. 2(1), impurities are those on vertical or horizontal edges.

We note that the dimer covering of $G$ is equivalent to the tiling of the corresponding region by dominoes and impurities (Fig. 2(2)).

$e \in M \cap \mathcal{E}_1$ connects vertices between $V(G) \cap (\mathcal{V}_1 \cup \mathcal{V}_2)$ and $V(G) \cap \mathcal{V}_3$ while $e \in M \cap \mathcal{E}_2$ connects those of $V(G) \cap (\mathcal{V}_1 \cup \mathcal{V}_2)$. Therefore the number of impurities is constant and given by

$$\sharp\{\text{impurities}\} = \frac{|V(G) \cap \mathcal{V}_1| + |V(G) \cap \mathcal{V}_2| - |V(G) \cap \mathcal{V}_3|}{2}. \tag{1.1}$$

*Remark 1.1* In this paper, it is always assumed that the RHS of (1.1) is a non-negative integer. If the locations of impurities are fixed, then this problem becomes a special case of the dimer-monomer problem where many results are known (e.g., [7]). Our graph $G$, with different boundary condition, has been introduced and called the Aztec rectangle with extra edges, in the new problem section of a review article [17] (Fig. 7). The Aztec rectangle with extra edges is not studied in this paper, but it can be regarded as $G^{(m,n)}$ given below for suitable $m, n$ with some impurities being fixed (Remark 4.3).

We next introduce two classes of finite subgraphs of $\mathcal{G}$ studied in this paper.

(1) $G^{(m,n)}$
$G^{(m,n)}(\subset \mathcal{G})$ is the rectangle which has $m$-blocks in the horizontal direction and $n$-blocks in

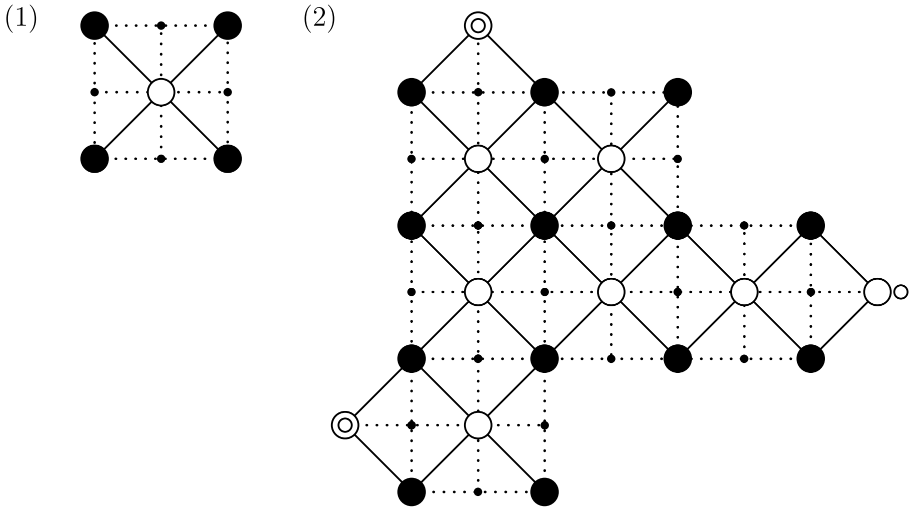



**Fig. 3** (**1**) The basic block. (**2**) An example of $G^{(2)}$. The double circles are the terminals. We have rotated $\mathcal{G}$ by $\frac{\pi}{4}$

the vertical direction. Figure 2(1) shows the case of $(m,n)=(3,2)$. Substituting $|V(G)\cap \mathcal{V}_1|+|V(G)\cap \mathcal{V}_2|=(m+1)(n+1)$, $|V(G)\cap \mathcal{V}_3|=mn$ to (1.1), we see that the number of impurities is equal to $(m+n+1)/2$ and the parity of $m$ and $n$ must be opposite.

(2) $G^{(k)}$
$G^{(k)}$ is made by the following procedure: (i) rotate $\mathcal{G}$ by $\frac{\pi}{4}$, (ii) compose arbitrarily by juxtaposition the "basic block" in Fig. 3(1) so that it is simply connected, and (iii) attach $(2k-1)$ vertices of $\mathcal{V}_1$ which we call "terminals" to the boundary. The number of impurities is equal to $k$. An example of $G^{(2)}$ is given in Fig. 3(2).

We consider two kinds of local move operation: square-move (s-move) and triangular-move (t-move), which transform a dimer covering to another one (Fig. 4).

In Sect. 3, we show the local move connectedness (LMC in short), i.e., any two dimer coverings of $G^{(m,n)}$, $G^{(k)}$ can be transformed each other by successive applications of local moves.

By LMC, we can simulate an ergodic Markov chain whose state space is

$$\mathcal{D}(G) := \{\text{dimer coverings of } G\}.$$

Under a suitable choice of transition probabilities, its stationary distribution is uniform so that we can obtain (approximately) uniform sample. Carrying out the simulation, we observe that the impurities are always attracted to the boundary of $G$, whichever the initial state is. Figures 5 and 8 show the results for $G^{(60,41)}$, and for a particular $G^{(20)}$ respectively, presented in terms of the equivalent tiling problem (Fig. 2(2)).

Thus we are led to the following conjecture which is our motivation to consider this problem.[1]

[1] Let $D$ be the number of dimer coverings of $G$, and let $C$ be the number of dimer coverings with an impurity being on the center of $G$. For $G=G^{(3,2)}$, $D=160$, $C=8$, and for $G=G^{(4,3)}$, $D=12400$, $C=400$.

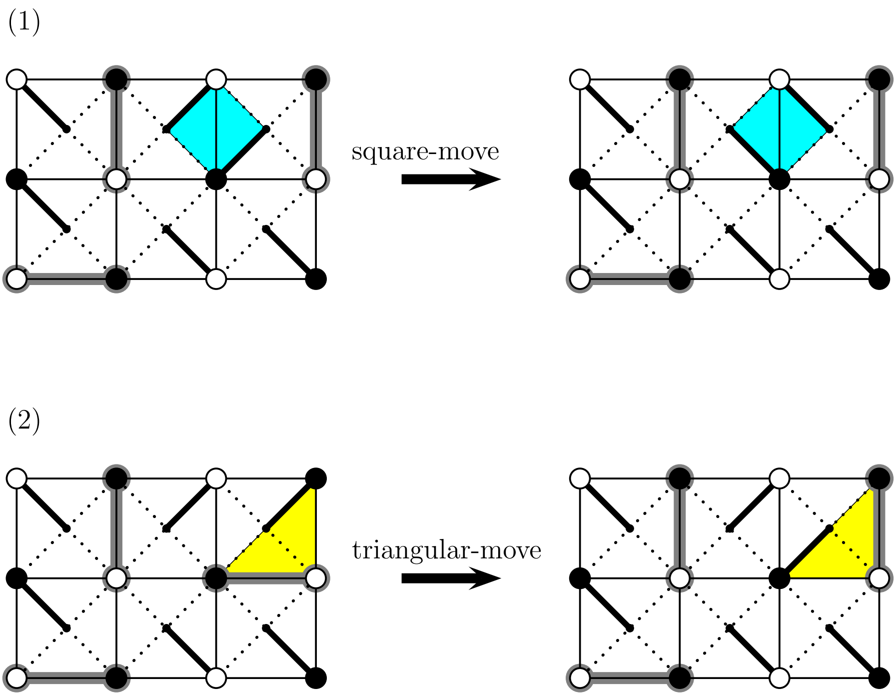



**Fig. 4** (**1**) Square move (s-move), (**2**) triangular move (t-move)

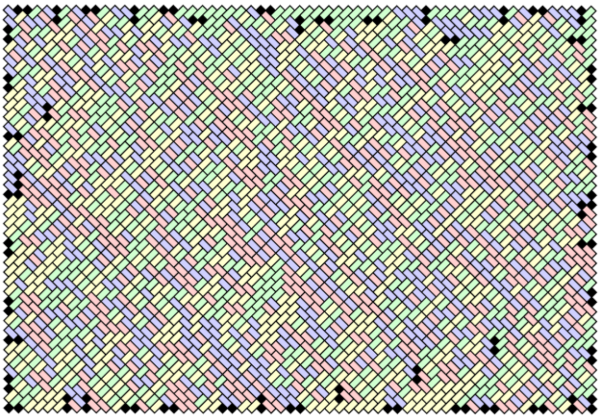


**Fig. 5** A result of simulation for $G^{(60,41)}$, presented in terms of the equivalent tiling problem. Impurities are attracted to boundaries

**Conjecture 1.2** *For $G = G^{(m,n)}$, $G^{(k)}$, the number of dimer coverings with given configuration of impurities is maximized if all impurities are on the boundary of $G$.*

Figure 6 shows the result of simulation for a Aztec rectangle with extra edges (Remark 1.1, Fig. 7) of size $60 \times 40$, where the flavor of the arctic circle theorem can be seen. One may notice that the direction of dominoes in the melting region is strongly influenced by the

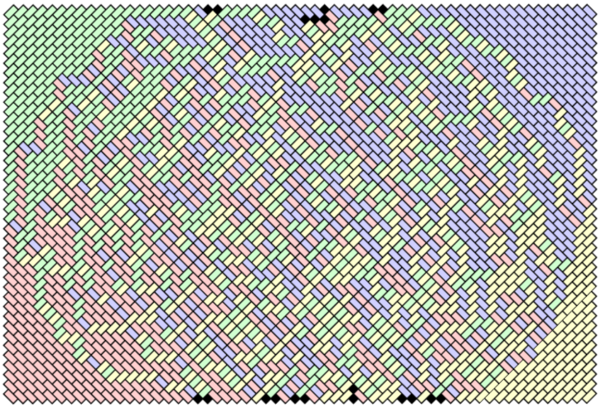


**Fig. 6** A result of simulation for an Aztec rectangle with extra edges (width $= 60$, height $= 40$)

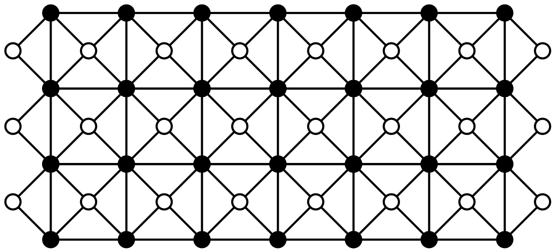


**Fig. 7** The Aztec rectangle with extra edges of size $7 \times 3$

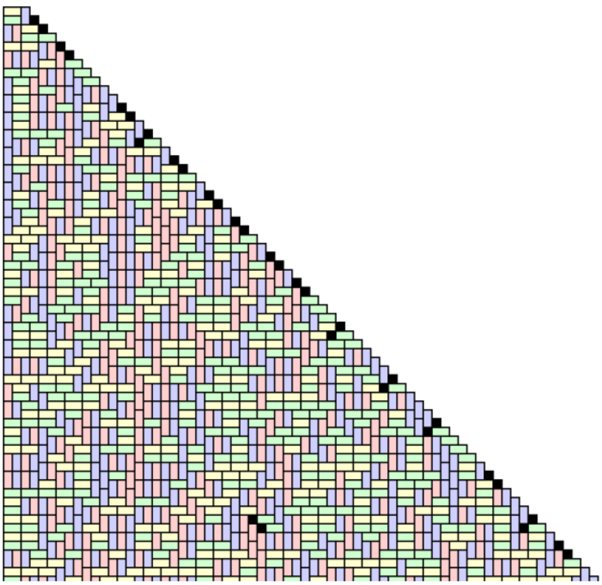


**Fig. 8** A result of simulation for $G^{(20)}$. All terminals are put on the oblique side

boundary as in the case of the Aztec diamond [5], whereas, in Fig. 5, dominoes away from the boundary seem to be free from the boundary effect.

In our previous work [14], we studied the case of 1-impurity and obtained a formula to compute $|\mathcal{D}(G^{(1)})|$ and the probability of finding the impurity at given edge $e$, which, if $G^{(1)}$ is the 1-dimensional chain, decreases exponentially as $e$ being far from the terminal of $G^{(1)}$ and thus Conjecture 1.2 is true for $G^{(1)}$ in the special case of a chain. In Sect. 4, we partially extend the analysis given in [14] to $G^{(k)}$.

As for related works, Ciucu [3, 4] studied the dimer-monomer problem on the hexagonal lattice and found that monomers interact each other as if they were the charged particles in 2-dimensional electrostatics. His graph and method are different from ours, but the same point of view may also be employed to our model.

The rest of this paper is organized as follows. In Sect. 2, we consider graphs $G_1$, $G_2$ with $V(G_1) = V(G) \cap \mathcal{V}_1$, $V(G_2) = V(G) \cap \mathcal{V}_2$ and show that there is a bijection between $\mathcal{D}(G)$ and the triples consisting of spanning forests of $G_1$, $G_2$ and the configuration of impurities satisfying certain condition (Theorems 2.1, 2.8). This bijection can be regarded as a generalization of the Temperley bijection [11] and gives us the global structure of the configuration of dimers. Such kind of relation is also known to hold for the Abelian sandpile model [6]. In Sect. 3, we prove LMC by using Theorem 2.8. LMC has been proved by [16] for normal subgraphs of $\mathcal{G}$. Our proof works only for $G^{(m,n)}$ and $G^{(k)}$ but is elementary and straightforward. It is possible to obtain a bound for the number of steps needed to transform two given dimer coverings each other, which depends polynomially on the volume of $G$. Thus it would be interesting to study the mixing time of the Markov chain discussed above. In Sect. 4, we extend the result in [14] to the case of $G^{(k)}$ ($k \geq 2$). We do not have exact formulas for $|\mathcal{D}(G^{(k)})|$ and the probability of finding impurities, but have bounds for them. In 1-dimensional case or the graph is small, this bound gives reasonable answer to Conjecture 1.2 (Sect. A.6). If the graph is large, however, the constant appearing in the inequality is too large to have a good quantitative understanding of Conjecture 1.2. This is because we ignore the interaction between impurities, the study of which should be done in a future work. On the other hand, from the argument in Sect. 4 it is possible to understand Conjecture 1.2 heuristically. In fact, in the graph $G^{(k)}$ it is reasonably clear that impurities are attracted to the terminals. The behavior of impurities in $G^{(m,n)}$ and the Aztec rectangles with extra edges are understood that some terminals are "hidden" on some part of the boundary (Remark 4.3). Since $G$ is planar, Kasteleyn's Theorem [9] gives $|\mathcal{D}(G)|$ in terms of the Pfaffian of a hopping matrix on $G$ with appropriate magnetic flux. However, this expression seems not to be suitable for our purpose to study Conjecture 1.2. In Sect. 5, we discuss some implications of Theorems 2.1, 2.8, and an application of Theorem 4.1 to 1-dimensional chain.

It is desirable to apply the argument in this paper to other graphs which is briefly discussed in Sect. A.5. It would also be interesting to consider the high dimensional version of this problem, yet for which the method in this paper is not applicable.

## 2 Transform to the Spanning Forest

In this section we construct a mapping from the set of dimer covering of $G$ to the set of spanning forests of two graphs made from $G$. We do this for $G = G^{(m,n)}$ in the Sect. 2.1, and for $G = G^{(k)}$ in the Sect. 2.2. We omit the superscript and write $G$ instead of $G^{(m,n)}$ in Sect. 2.1 (resp. instead of $G^{(k)}$ in Sect. 2.2). We first set

$$\begin{aligned} &V_1 := V(G) \cap \mathcal{V}_1, && V_2 := V(G) \cap \mathcal{V}_2, && V_3 := V(G) \cap \mathcal{V}_3, \\ &E_1 := E(G) \cap \mathcal{E}_1, && E_2 := E(G) \cap \mathcal{E}_2, && G = G^{(m,n)}, G^{(k)}. \end{aligned}$$

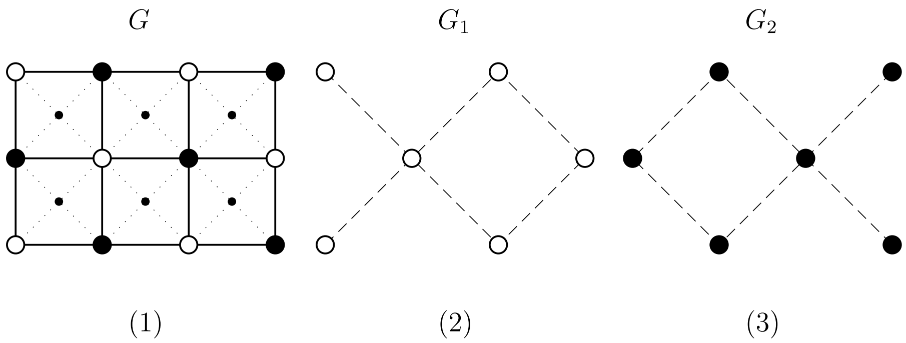



**Fig. 9** (**1**) $G^{(3,2)}$ (**2**), (**3**) $G_1$, $G_2$ corresponding to $G^{(3,2)}$

## 2.1 Construction of a Bijection: for $G^{(m,n)}$

Let $G_1$, $G_2$ be graphs such that $V(G_j) = V_j$ $(j = 1, 2)$ and for $x, y \in V_j$, we set $(x, y) \in E(G_j)$ iff we find $z \in V_3$, which we call a *middle vertex*, with $(x, z), (z, y) \in E_1$. An explicit description is given in Fig. 9.

Let $k := \frac{m+n+1}{2}$ be the number of impurities.

**Theorem 2.1** *We have a bijection between the following two sets.*

$$\begin{aligned}\mathcal{D}(G) &:= \{\textit{dimer coverings on } G\}\\ \mathcal{F}(G, P) &:= \{(F_1, F_2, \{e_j\}_{j=1}^k) \mid F_j\textit{: spanning forests on } G_j \ (j = 1, 2)\\ &\qquad\qquad \textit{with } k\textit{-components},\\ &\qquad\qquad \{e_j\}_{j=1}^k \subset E_2\textit{: configuration of impurities},\\ &\qquad\qquad \textit{with condition } \mathbf{(P)}\}\end{aligned}$$

**(P)**

(1) $F_1$, $F_2$ *have no intersections*
(2) *the trees of* $F_1$ *are paired with those of* $F_2$ *by impurities*

Under condition (P)(1), a spanning forest of $G_1$ uniquely determines that of $G_2$ so that we have a redundancy in the statement. Figure 10(1) shows an example of spanning forests and impurities with condition (P) for $G^{(3,2)}$. In this figure, spanning trees of both $G_1$ and $G_2$ are composed of three trees for each and are paired by impurities. Figure 10(2) is the corresponding dimer covering of $G$.

In Appendix A, we compute $|\mathcal{D}(G^{(1,2k)})|$ by using Theorem 2.1.

To prove Theorem 2.1, we construct a map from $\mathcal{D}(G)$ to $\mathcal{F}(G, P)$ and its inverse, after some preparations. First of all, given a dimer covering of $G$, we draw curves on $G$, which we call *slit curves* [14], as is described in Figs. 11, 12.

We study some properties of these curves. Some inspections lead us to the following observation.

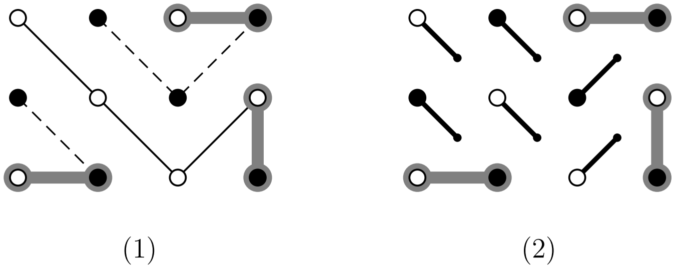



**Fig. 10** (**1**) An example of spanning forests and impurities with condition (P). *Solid lines* indicate a forest of $G_1$, and *dotted lines* indicate a forest of $G_2$. (**2**) The corresponding dimer covering of $G$

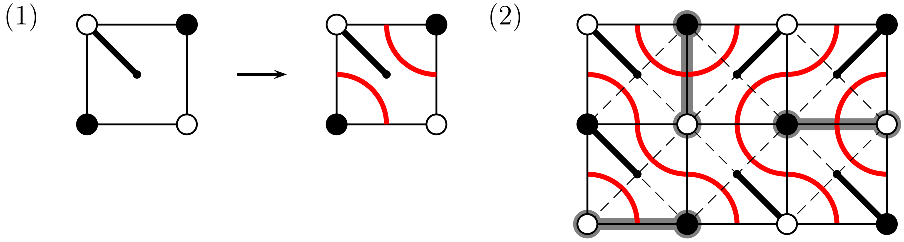



**Fig. 11** (**1**) Drawing slit curves on a unit block in $G$ with a given dimer covering. (**2**) The slit curves corresponding to the dimer covering in Fig. 2(1)

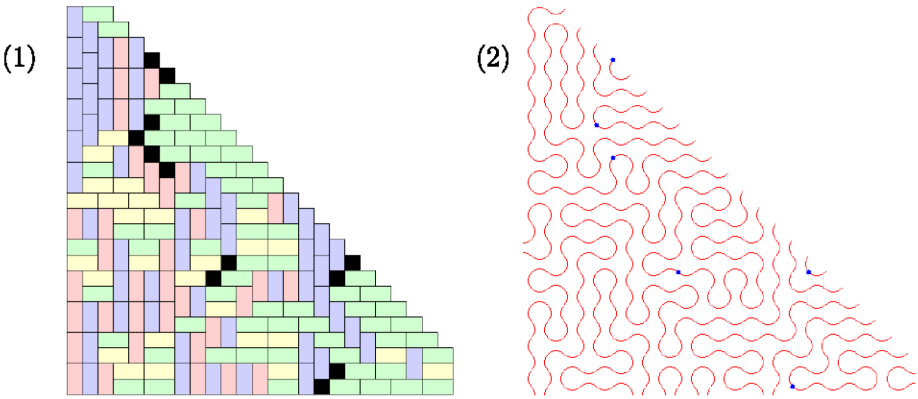



**Fig. 12** An example of dimer covering on $G^{(6)}$ and the corresponding slit curves

**Proposition 2.2** (1) *By a triangular-move, impurities move along slit curves, but slit curves remain unchanged.*
(2) *By a square-move, slit curves may change, but impurities remain unchanged.*

Figure 13 explains what they mean by presenting an explicit example.

These slit curves divide $G$ into some subgraphs, which we call domains. Vertices in $V_1$ and those in $V_2$ are not in the same domain so that, by ignoring middle vertices, these

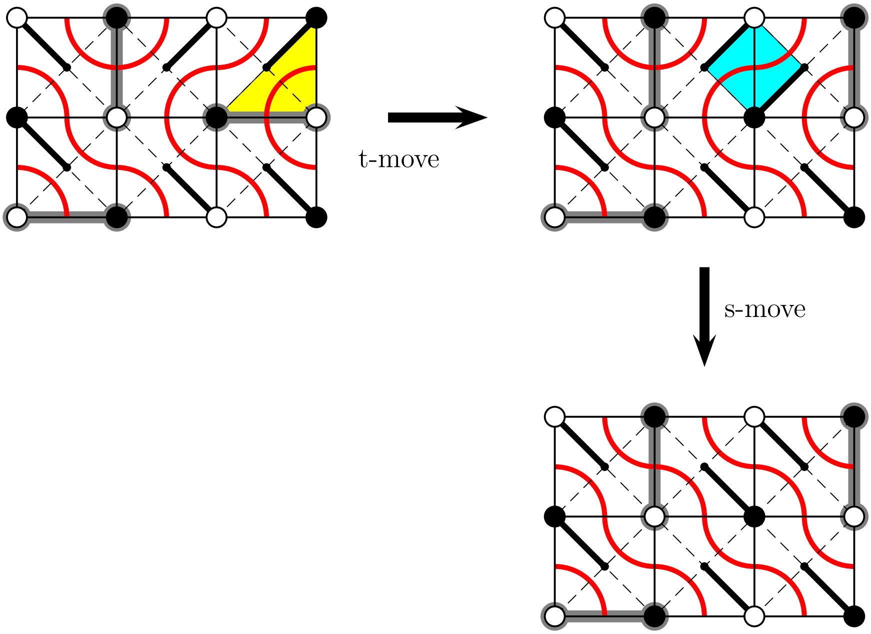
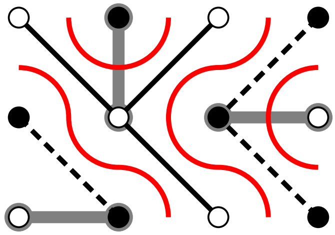



**Fig. 13** Changes of impurities and slit curves under local moves

**Fig. 14** Each impurity is always penetrated by a slit curve at their middle and thus lives in two neighboring domains



domains can be regarded as subgraphs of $G_1$, $G_2$. Each impurity is always penetrated by a slit curve at their middle and thus lives in two neighboring domains. As an example, Fig. 14 shows the domains corresponding to the dimer covering in Fig. 11(2).

**Proposition 2.3**

(0) *Slit curves do not branch and do not terminate inside* $G$.
(1) *Slit curves are not closed.*
(2) *The number of vertices in each domain* (*as a subgraph of* $G$) *is always odd.*
(3) *Impurities always make pairings between two neighboring domains.*

(0) and (1) implies that every slit curve terminates at boundary. Figure 15 shows an example of (3).

*Proof* (0) is clear.

(1) It is the consequence of Proposition 2 in [14].

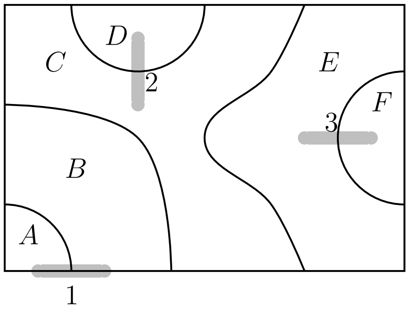



**Fig. 15** An outline of Fig. 11(2). The impurity 1 connects domains $A$, $B$, the impurity 2 connects domains $C$, $D$ and the impurity 3 connects domains $E$, $F$

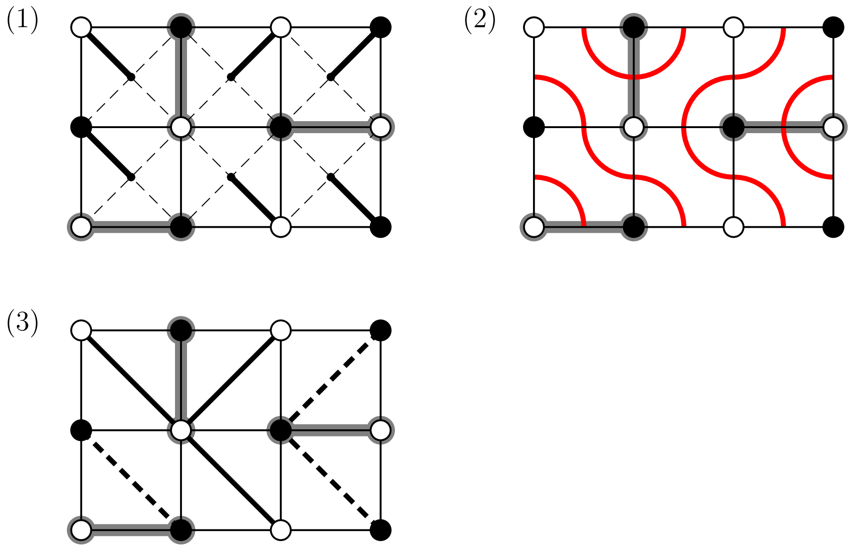



**Fig. 16** (**1**) A dimer covering, (**2**) the corresponding configurations of slit curves and impurities, and (**3**) the corresponding spanning forests of $G_1$ and $G_2$ with condition $(P)$. *Thick lines* indicate a forest of $G_1$, and *dotted lines* indicate a forest of $G_2$

(2) By (1), each domain is a tree. It then suffices to note that, when we add a unit block[2] to a domain, the number of vertices in this domain increases by two.

(3) The number of impurities is equal to $\frac{m+n+1}{2}$, while that of domains is equal to $m + n + 1$, because we have no closed curves. By (2), we must have odd number of impurities in each domain. Since the number of impurities is half of that of domains, each domain has one impurity, hence impurities have to make pairings between domains. □

By Proposition 2.3(1), each domain, being regarded as a subgraph of $G_1$ or $G_2$, is a tree so that we obtain spanning forests $F_1$, $F_2$ of $G_1$, $G_2$. Moreover, by Proposition 2.3(3), each tree of $F_1$, $F_2$ is paired by impurities so that we obtain an element $(F_1, F_2, \{e_j\}_{j=1}^k) \in \mathcal{F}(G, P)$. Figure 16 is a flowchart of our discussion.

Thus it suffices to construct the inverse mapping to finish the proof of Theorem 2.1.

[2]Unit block is the one described in Fig. 11(1).

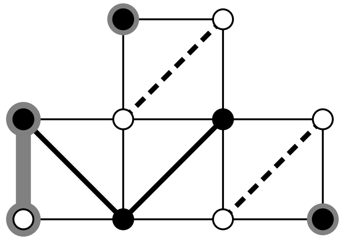


**Fig. 17** Example of spanning forest in general case

**Proposition 2.4** *For a given element* $(F_1, F_2, \{e_j\}_{j=1}^k) \in \mathcal{F}(G, P)$, *we can find the corresponding dimer covering uniquely.*

*Proof* It clearly suffices to find the dimer covering on each tree of spanning forests, being regarded as a subgraph of $G$ by adding middle vertices. Each tree is further divided into a number of subtrees by impurities, where the numbers of vertices are always even. It remains to make the dimer covering on each tree, regarding the impurity as the root. □

The proof of Theorem 2.1 is completed.

*Remark 2.5* Let $(F_1, F_2, \{e_j\}_{j=1}^k) \in \mathcal{F}(G, P)$ and let $\mathcal{T} = \{T_j\}_{j=1}^{2k}$ be the set of trees which compose $F_1, F_2$. We can regard $\mathcal{T}$ as a bipartite graph, by setting $T_i, T_j$ $(i \neq j)$ are adjacent iff $T_i, T_j$ share a slit curve in their neighbor. Then a configuration of impurities $\{e_j\}$ with condition (P) is identified with a dimer covering on $\mathcal{T}$.

*Remark 2.6* Theorem 2.1 also works for graphs which are made by composing the unit block arbitrarily, provided it is simply connected and the circumference $L$ of that satisfies $L \in 4\mathbf{N} + 2$, in which case the number of impurities is equal to $\frac{L+2}{2}$ (Fig. 17).

### 2.2 Construction of a Bijection: for $G^{(k)}$

In this subsection, we set $G := G^{(k)}$ and label its terminals as $T_1, T_2, \ldots, T_k$. To state our theorem, we need numerous notations which are introduced here.

**Notation 1** (1) As is done for $G^{(m,n)}$, let $G_j$ $(j = 1, 2)$ be the graph such that $V(G_j) = V_j$, and for $x, y \in V_j$, we set $(x, y) \in E(G_j)$ iff there is $z \in V_3$, which we call *middle vertex*, with $(x, z), (z, y) \in E_1$. Figure 18 shows $G_1, G_2$ for the example given in Fig. 3(2). Putting back the middle vertices on $G_1, G_2$ yields subgraphs of $G$ which we call $G_1', G_2'$ (Fig. 19).

(2) Let $x, y$ be vertices. We say that $x$ is *directly connected* to $y$ iff we have the edge $e = (x, y)$. We say $x$ is a *boundary vertex* iff $x \in V_3$ and lies on the boundary of $G$.

(3) Let $\overline{G}$ be the graph obtained from $G$ by the following procedure: (i) add an imaginary vertex $R$ which we call the *root*, and (ii) connect all terminals and boundary vertices directly to $R$. We call edges of the form $e = (R, y)$ *outer edges*. $\overline{G}$ for the example in Fig. 3(2) is shown in Fig. 20.

(4) Let $\overline{G_1}$ be the graph such that $V(\overline{G_1}) = V_1 \cup \{R\}$, and for $x, y \in V(\overline{G_1})$ we set[3] $(x, y) \in E(\overline{G_1})$ iff $x = R$, $y \in \{T_j\}_{j=1}^k$, or if there is $z \in V_3$ with $(x, z), (z, y) \in E(\overline{G})$. $\overline{G_1}$

[3]We allow multiple edges.

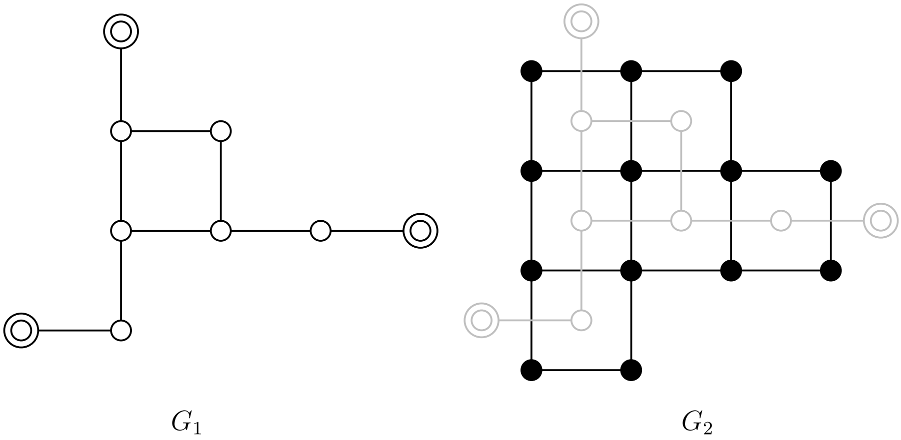



**Fig. 18** $G_1, G_2$ corresponding to the example in Fig. 3(2)

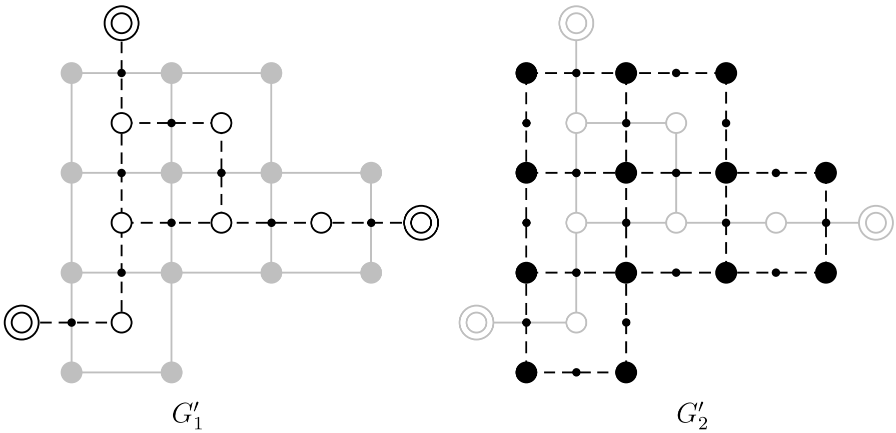



**Fig. 19** $G_1', G_2'$ corresponding to the example in Fig. 3(2)

for the example in Fig. 3(2) is given in Fig. 21(1). Putting back middle vertices on $\overline{G_1}$ yields a subgraph $\overline{G_1'}$ of $\overline{G}$ (Fig. 21(2)).

*Remark 2.7* For a subgraph $A(\subset \overline{G_1})$ of $\overline{G_1}$, its edge $e = (x, y) \in E(A)$ contains a vertex of $V_3$ at its middle (except $e$ connects a terminal and $R$). Adding such middle vertices yields a subgraph $A'$ of $\overline{G_1'}$. We always identify $A$ with $A'$ and thus regard $A$ as a subgraph of $\overline{G_1'}$. Conversely, for a subgraph $A'(\subset \overline{G_1'})$ of $\overline{G_1'}$, ignoring middle vertices from $A'$ yields a subgraph $A$ of $\overline{G_1}$. Similarly, cutting outer edges from $A'$, we obtain a subgraph $\tilde{A}$ of $G_1'$. In both cases, we identify $A'$ with $A$ or $\tilde{A}$, and regard $A'$ as a subgraph of $\overline{G_1}$ or $G_1'$.

To state an analogue of Theorem 2.1, we further need some notations.

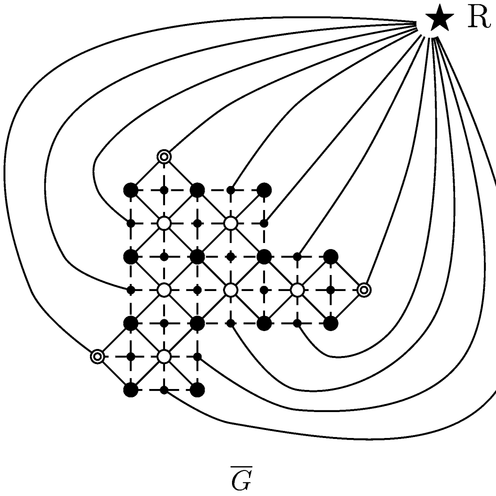



**Fig. 20** $\overline{G}$ for the example in Fig. 3(2)

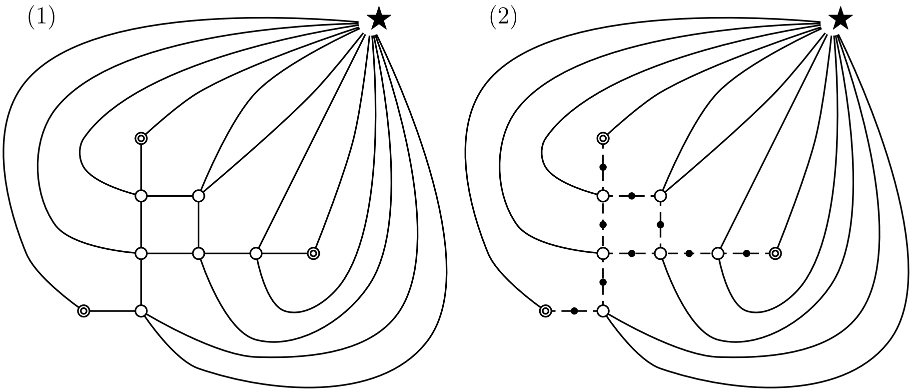



**Fig. 21** (**1**) $\overline{G_1}$, (**2**) $\overline{G_1'}$ for the example in Fig. 3(2)

**Notation 2** Let $A(\subset \overline{G_1'})$ be a subgraph(not necessarily a tree) of $\overline{G_1'}$.

(1) We say that $A$ is a *TI-tree* iff (i) $A$ is a tree, (ii) $A$ contains a unique terminal, and that terminal is connected directly to $R$, and (iii) $A$ has no boundary vertices (Fig. 22).

(2) We say that $A$ is a *TO-domain* iff (i) $A$ is a tree when restricted to $G_1'$, (ii) $A$ contains a unique terminal, and that terminal is not connected directly to $R$, and (iii) $A$ has boundary vertices, and all of them are connected directly to $R$ (Fig. 23).

We say a TO-domain $A$ is a *TO-tree* if it is a tree (and so for other ones below), which means boundary vertex is unique (Fig. 24).

(3) We say that $A$ is a *IO-domain* iff (i) $A$ is a tree when restricted to $G_1'$, (ii) $A$ is not connected to terminals, and (iii) $A$ has boundary vertices all of which are connected directly to $R$ (Fig. 25). We say a IO-domain $A$ is a *IO-tree* if it is a tree.

(4) Let $l = 1, 2, \ldots$. We say that $A$ is a $T^l I$*-tree* iff (i) $A$ is a tree, (i) $A$ contains $l$-terminals $T_{i_1}, T_{i_2}, \ldots, T_{i_l}$, $i_1 < i_2 < \cdots < i_l$, among which only $T_{i_1}$ is connected directly to $R$, and (ii) $A$ has no boundary vertices (Fig. 26).

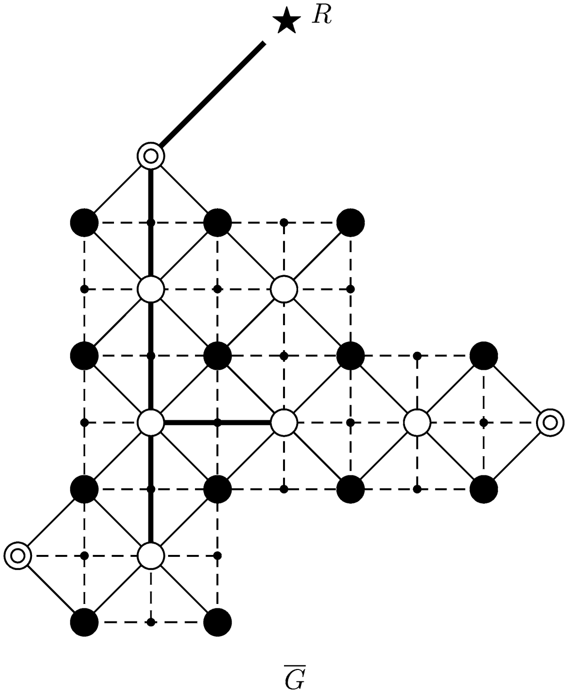



**Fig. 22** An example of TI-tree

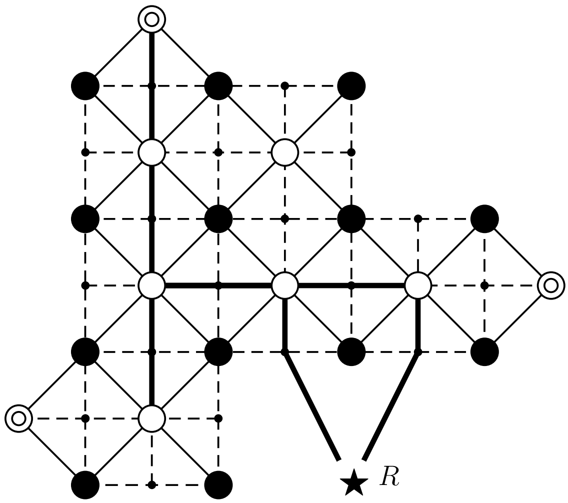



**Fig. 23** An example of TO-domain

(5) We say that $A$ is a $T^l O$*-domain* iff (i) $A$ is a tree when restricted to $G_1'$, (ii) $A$ contains $l$ terminals $T_{i_1}, T_{i_2}, \dots, T_{i_l}$, and all of them are not connected directly to $R$, and (iii) $A$ has boundary vertices, and all of them are connected directly to $R$ (Fig. 27).

(6) Let $T$ be a spanning tree of $\overline{G_1}$. If we cut $T$ at the root $R$, we would have a number of trees, say $A_1, A_2, \dots, A_n$, which are all connected to $R$. In this case we say that $T$ is *composed of* $A_1, A_2, \dots, A_n$.

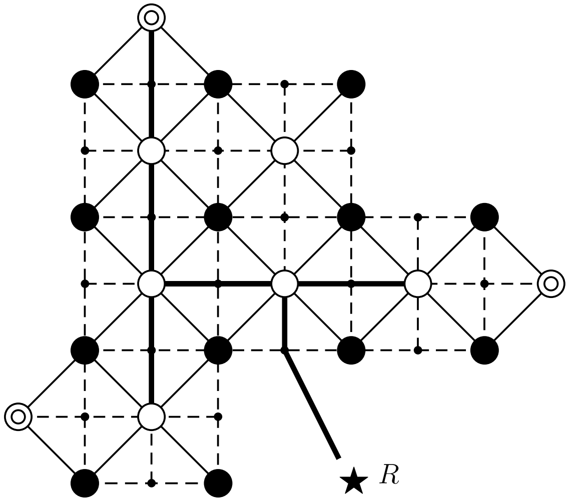



**Fig. 24** An example of TO-tree

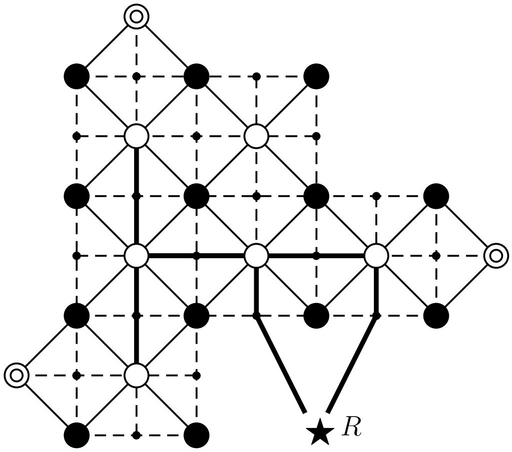



**Fig. 25** An example of IO-domain

**Theorem 2.8** *We have a bijection between the following two sets.*

$$\mathcal{D}(G) := \{\textit{dimer coverings of } G\}$$

$$\mathcal{F}(G, Q) := \Big\{\big(T, S, \{e_j\}_{j=1}^k\big) \mid T\textit{: spanning tree of } \overline{G_1},\ S\textit{: spanning forest of } G_2,$$
$$\{e_j\}_{j=1}^k \subset E_2\textit{: configuration of impurities, with condition } \mathbf{(Q)}\Big\}$$

**(Q)**

(1) *$T$ is composed of $k$ TI-trees, $(k-1)$ TO-trees, and the other ones are IO-trees.*
(2) *$S$ is composed of $k$ trees.*
(3) *$T$, $S$ are disjoint of each other, and the $k$ TI-trees of $T$ and the $k$ trees of $S$ are paired by impurities.*

Figures 28 and 29 show an example and the corresponding dimer covering.

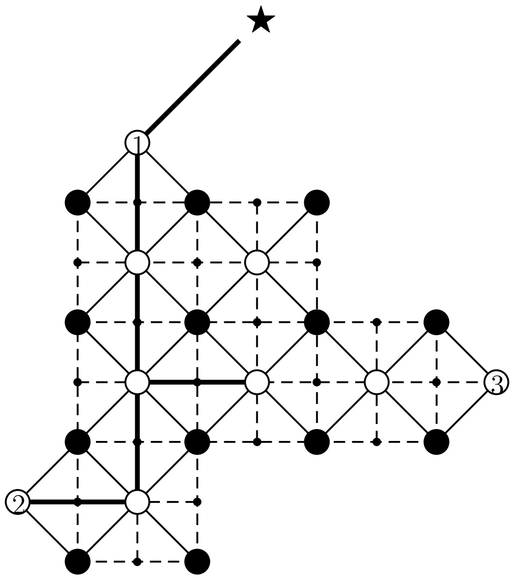



**Fig. 26** An example of $T^2 I$-tree. Terminals with smallest index is connected directly to $R$

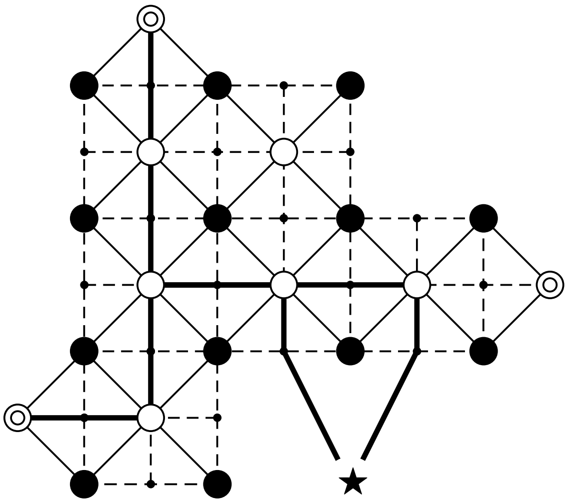



**Fig. 27** An example of $T^2 O$-domain

*Remark 2.9* (1) A spanning tree $T$ of $\overline{G_1}$ uniquely determines a spanning forest $S$ of $G_2$ under the condition that they are disjoint. (2) In condition (Q)(1), $T^l I$-trees are counted as one TI-tree and $(l-1)$ TO-trees (Fig. 30), and $T^l O$-domains are counted as $l$ TO-domains (Fig. 31, for the graph $G^{(3)}$).

*Proof* As in the proof of Theorem 2.1, it suffices to construct mappings $T_{\mathrm{FD}} : \mathcal{F}(G, Q) \to \mathcal{D}(G)$ and $T_{\mathrm{DF}} (= T_{\mathrm{FD}}^{-1}) : \mathcal{D}(G) \to \mathcal{F}(G, Q)$. For $T_{\mathrm{FD}} : \mathcal{F}(G, Q) \to \mathcal{D}(G)$, we note that, as subgraphs of $G_1'$, $G_2'$, the numbers of vertices of a TI-tree and trees in $G_2'$ are odd, while those of a TO-tree and a IO-tree are even. Putting impurities, they become all even, so that it

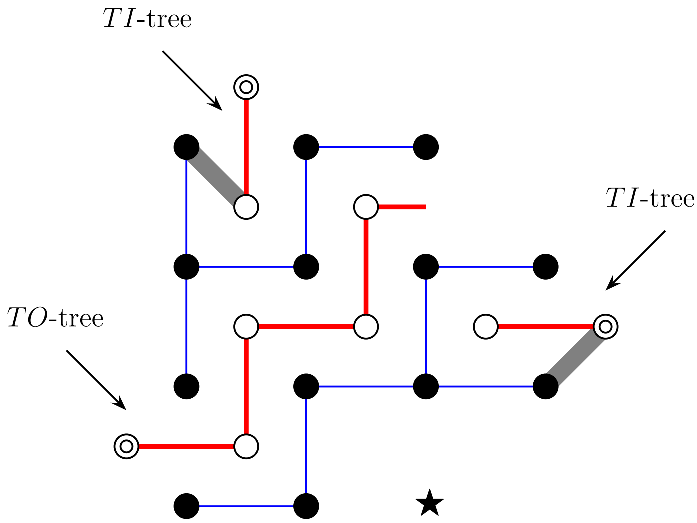



**Fig. 28** An illustration of Theorem 2.8. *Thick lines* indicate a spanning tree $T$ of $\overline{G_1}$, and *thin lines* indicate a spanning forest $S$ of $G_2$

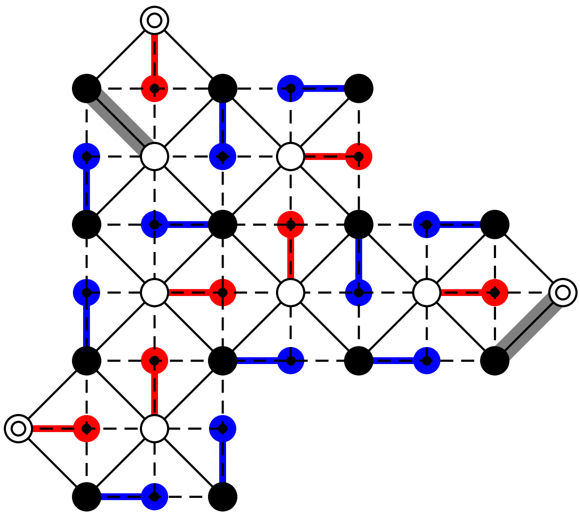
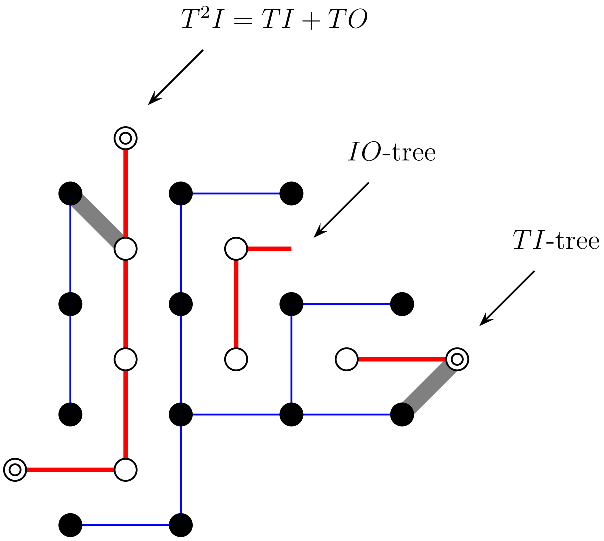


**Fig. 29** The corresponding dimer covering

suffices to find the corresponding dimer covering on each tree by the argument in the proof of Proposition 2.4.

It then suffices to construct a mapping $T_{\mathrm{DF}} : \mathcal{D}(G) \to \mathcal{F}(G, Q)$. Given a dimer covering on $G$, we draw the corresponding slit curves as is done in Fig. 11, which divide $G$ into some domains. Since vertices in $V_1$ and those in $V_2$ are not in the same domain, each domain can be regarded as a subgraph of either $G_1'$ or $G_2'$. By Proposition 2 in [14], slit curves are not closed, so that domains in $G_1'$ are either TI-tree, TO-domain, $T^l I$-tree, $T^l O$-domain, or IO-domain. Moreover, domains in $G_2'$ are trees. Here we regard those trees and domains as subgraphs in $G_1'$, $G_2'$ (Remark 2.7). In what follows, as is already mentioned, we regard

**Fig. 30** $T^2I$-tree is counted as one *TI*-tree and one *TO*-tree



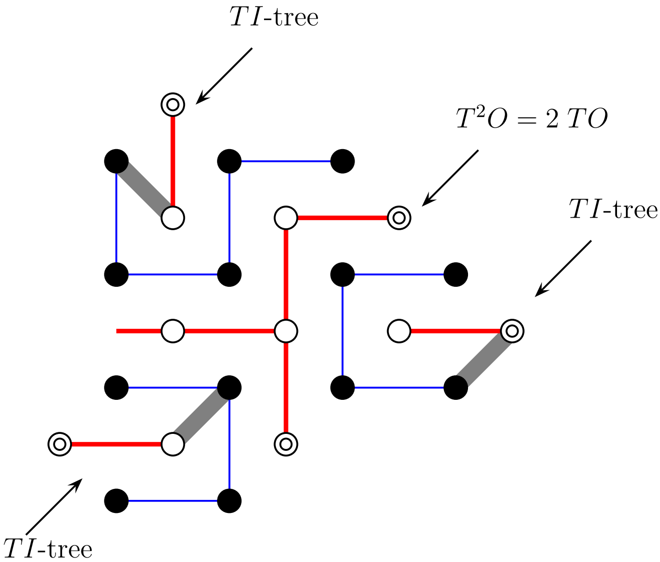




**Fig. 31** $T^2O$-tree is counted as two *TO*-trees

(count) a $T^lI$-tree as a composition of a TI-tree and $(l-1)$ TO-trees, and regard a $T^lO$-domain as $l$ TO-domains (Remark 2.9). We then note the following facts.

(i) Because the numbers of vertices in TI-trees and trees in $G'_2$ are odd, they should have impurities.
(ii) If $\sharp\{\text{TO-domains}\} = l$, then $\sharp\{\text{trees in } G'_2\} \geq l+1$.
(iii) $\sharp\{\text{TI-trees}\} + \sharp\{\text{TO-domains}\} = 2k-1$.

Using these facts, we proceed

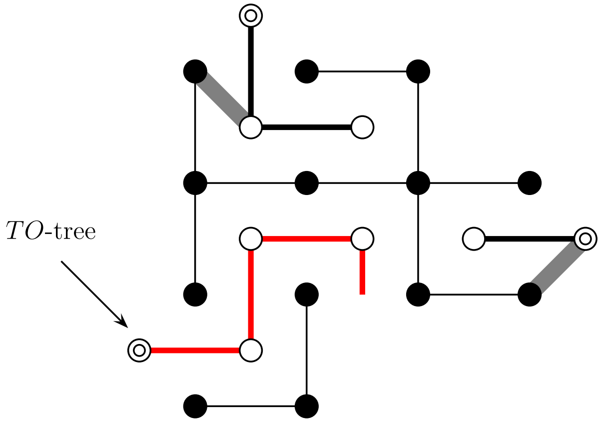



**Fig. 32** The boundary vertex of TO-tree must be located farther than the next terminal, in order that the pairing condition Q(3) is satisfied

(a) Since we have $k$ impurities, by (i) $\sharp\{\text{TI-trees}\} \le k$, so that by (iii) $\sharp\{\text{TO-domains}\} \ge k-1$.
(b) Since $\sharp\{\text{trees in } G_2'\} \le k$, by (ii) we should have $\sharp\{\text{TO-domains}\} \le k-1$ so that by (iii) $\sharp\{\text{TI-trees}\} \ge k$.

By (a), (b), it follows that $\sharp\{\text{TI-trees}\} = k$, $\sharp\{\text{TO-domains}\} = k-1$, and thus TI-trees and trees in $G_2'$ are paired by impurities. Each TO-domain and IO-domain has only one boundary vertex so that they are TO-trees and IO-trees respectively, since otherwise the number of trees in $G_2'$ would be larger than $k$.

We next connect the terminals of TI-trees directly to $R$, connect the boundary vertices of $T^l O$-trees directly to $R$, and connect directly to $R$ the terminal $T_{i_1}$ which has the smallest index among $T_{i_1}, \dots, T_{i_l}$ $(i_1 < i_2 < \cdots < i_l)$ of $T^l I$-trees. Then we obtain a spanning tree $T$ of $\overline{G_1}$. By the arguments above, the spanning tree $T$ of $\overline{G_1}$, the spanning forest $S$ of $G_2$ and the $k$ impurities satisfy the condition (Q). The proof of theorem 2.8 is completed. □

*Remark 2.10* The boundary vertex of TO-tree must be located farther than the next terminal, since otherwise the pairing condition Q(3) would not be satisfied (Fig. 32). Hence the mapping $T_{\mathrm{DF}} : \mathcal{D}(G) \to \mathcal{F}(G, Q)$ does not exhaust all spanning trees of $\overline{G_1}$.

When $k = 1$, we further identify the terminal with the root, and then the statement of Theorem 2.8 is simplified as follows.

**Corollary 2.11** *We have a bijection between the following two sets.*

$$\mathcal{D}(G) := \{\textit{dimer coverings in } G\}$$
$$\mathcal{T}\big(\overline{G_1}, Q'\big) := \big\{(T_1, e) \mid T_1\text{: \textit{spanning tree of} } \overline{G_1},\; e \in E_2\text{: \textit{location of impurity, with condition} } (\boldsymbol{Q'})\big\}$$

**(Q′)**
*If we regard the spanning tree of $\overline{G_1}$ as the spanning forest of $G_1$, the impurity lies on the tree containing the terminal.*

Figure 33 describes an example.

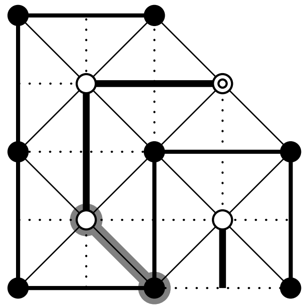


**Fig. 33** An example of Corollary 2.11

## 3 Local Move Connectedness

In this section we show that both $G^{(m,n)}$ and $G^{(k)}$ have LMC. Let $N := |V(G)|$ be the volume of $G$, and for $G = G^{(m,n)}$ let $k := (m+n+1)/2$ $(= O(N^{1/2})$ for large$N)$ be the number of impurities.

**Theorem 3.1** *For $G^{(m,n)}$, $G^{(k)}$, any two dimer coverings can be transformed each other by successive applications of local moves of $O(k^2 N^{3/2})$-steps.*

LMC is proved for normal subgraphs of $\mathcal{G}$ in [16], but our proof here is simpler. LMC for the triangular lattice is proved in [12].

*Proof* (1) We first show LMC for $G := G^{(k)}$. For any $\{e_j\}_{j=1}^k \subset E_2(G)$, let

$$\mathcal{D}\big(G; \{e_j\}_{j=1}^k\big) := \big\{M \in \mathcal{D}(G) \,|\, \text{impurities are on } \{e_j\}_{j=1}^k\big\},$$
$$\mathcal{D}_B(G) := \big\{M \in \mathcal{D}(G) \,|\, \text{all impurities are on the boundary}\big\}.$$

We denote by $E_2(G) \cap \partial G$ the set of edges on the boundary of $G$. Then we clearly have

$$\mathcal{D}_B(G) = \bigcup_{\{e_j\}_{j=1}^k \subset E_2(G) \cap \partial G} \mathcal{D}\big(G; \{e_j\}_{j=1}^k\big).$$

In any dimer covering, impurities can always be moved to the boundary by applying t-moves. In other words, for any dimer covering $M \in \mathcal{D}(G)$, we can find $M' \in \mathcal{D}_B(G)$ which is connected to $M$.[4] If we put all impurities on the boundary and fix them, say on $\{e_j\}_{j=1}^k \subset E_2(G) \cap \partial G$, then our dimer problem is reduced to that of the domino tiling on $G \setminus \{e_j\}_{j=1}^k$ (Fig. 34) which are connected via s-moves, in $O(N^{3/2})$-steps [13, 19], implying that elements in $\mathcal{D}(G; \{e_j\}_{j=1}^k)$ are connected each other for fixed $\{e_j\}_{j=1}^k \subset E_2(G) \cap \partial G$. It then suffices to show that an element in $\mathcal{D}(G; \{e_j\}_{j=1}^k)$ is connected to some element in $\mathcal{D}(G; \{e'_j\}_{j=1}^k)$ for any $\{e_j\}_{j=1}^k$, $\{e'_j\}_{j=1}^k \subset E_2(G) \cap \partial G$, unless $\mathcal{D}(G; \{e_j\}_{j=1}^k) = \emptyset$ or $\mathcal{D}(G; \{e'_j\}_{j=1}^k) = \emptyset$.

Here we rephrase the LMC for the domino tiling [13, 19] as follows: all possible configurations of spanning trees of $\overline{G_1}$ are connected via s-moves of $O(N^{3/2})$-steps, provided all impurities are fixed and are on the boundary.

[4] We say henceforth that $M, M' \in \mathcal{D}(G)$ are *connected* if they can be transformed via local moves.

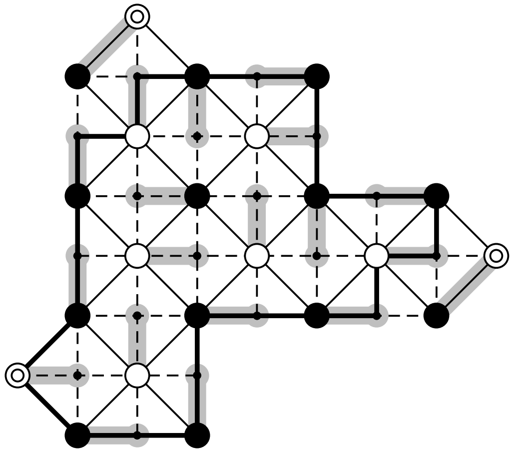


**Fig. 34** If we fix all impurities on $\{e_j\}_{j=1}^k \subset E_2(G) \cap \partial G$, then our dimer problem is reduced to that of the domino tiling on $G \setminus \{e_j\}_{j=1}^k$ (the region surrounded by *thick lines*)

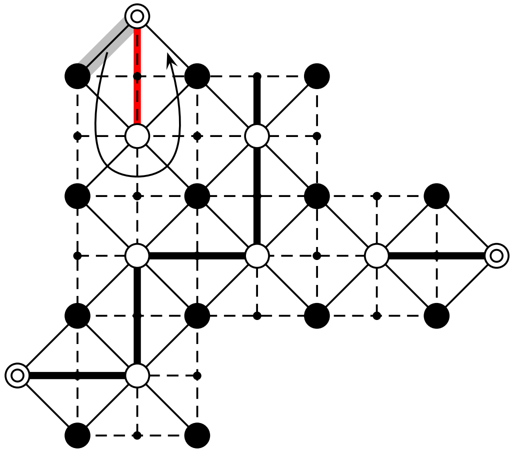


**Fig. 35** Each terminal has two positions, on which the impurity can be switched by making a TI-tree

For any $M \in \mathcal{D}_B(G)$, impurities can occupy two positions on each terminal. By making a TI-tree on this terminal, which is done in $O(N^{3/2})$-steps, the impurity on one of these two positions can be moved to the other one (Fig. 35).

On the other hand, the impurity on a terminal can be moved to the one in nearest neighbor by making a $T^2I$-tree between these two terminals, provided the nearest neighbor terminal does not have an impurity (Fig. 36).

Therefore, impurities can always be moved from a terminal to the vacant one next to it, and by repeating this procedure at most $O(k^2)$-steps, an element in $\mathcal{D}(G; \{e_j\}_{j=1}^k)$ is connected to some element in $\mathcal{D}(G; \{e'_j\}_{j=1}^k)$, in $O(k^2 N^{3/2})$-steps.

*Remark 3.2* It can happen that $\mathcal{D}(G; \{e_j\}_{j=1}^k) = \emptyset$ for some $\{e_j\}_{j=1}^k \subset E(G) \cap \partial G$. However, it is always possible to avoid such configuration of impurities in the argument above.

(2) We next show LMC for $G^{(m,n)}$. In fact, it is reduced to that for $G^{(k)}$ by embedding $G = G^{(m,n)}$ to $G' = G^{(k)}$, $k = \frac{m+n+1}{2}$ by attaching some extra vertices on the boundary (Fig. 37).

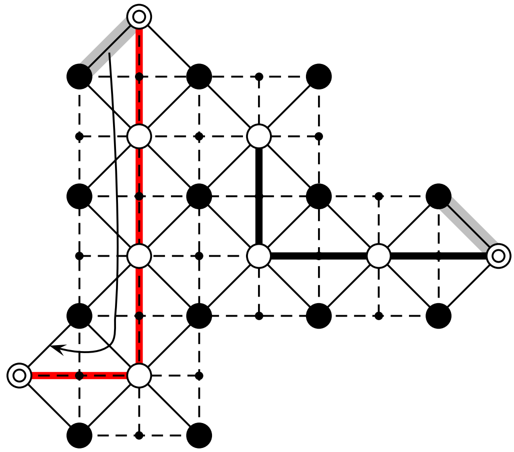


**Fig. 36** The impurity on a terminal can be moved to the neighbor by making a $T^2I$-tree between these two terminals

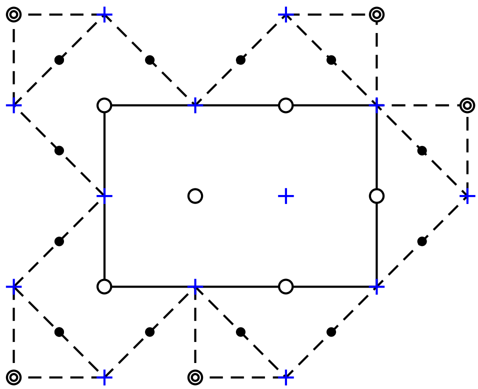


**Fig. 37** $G^{(m,n)}$ can be embedded to some $G^{(k)}$, $k = \frac{m+n+1}{2}$. *Solid lines* are boundary of $G^{(3,2)}$ and *dotted lines* are boundary of $G^{(3)}$

We specify the dimers covering the vertices $V(G') \setminus V(G)$, so that $M \in \mathcal{D}(G)$ can be regarded as $M' \in \mathcal{D}(G')$ (Fig. 38(1)). In particular, the bijection theorem for $G^{(k)}$ (Theorem 2.8) also applies for $G^{(m,n)}$.

Moreover, we can also see that, by looking at the slit curve corresponding to Fig. 38(1) explicitly, the configuration of impurities on the boundary for $G$ and that for $G'$ are in one to one correspondence (Fig. 38(2)). Hence by the argument in (1), we can prove LMC without moving dimers on $V(G') \setminus V(G)$, completing the proof of Theorem 3.1. □

*Remark 3.3* The argument in the proof of Theorem 3.1 as well as that in [16] also proves LMC for the Aztec rectangles with extra edges (Fig. 7).

## 4 Estimate for the Number of Dimer Coverings and Probability of Impurity Configuration

In [14], we studied one impurity case $G^{(1)}$, in which case dimer covering is given by spanning tree and location of the impurity (Corollary 2.11). The uniform distribution on the set of spanning trees is generated by the loop erased random walk [2], and the number of configurations of impurity depends only on the length of the tree. Thus, by using the theory of

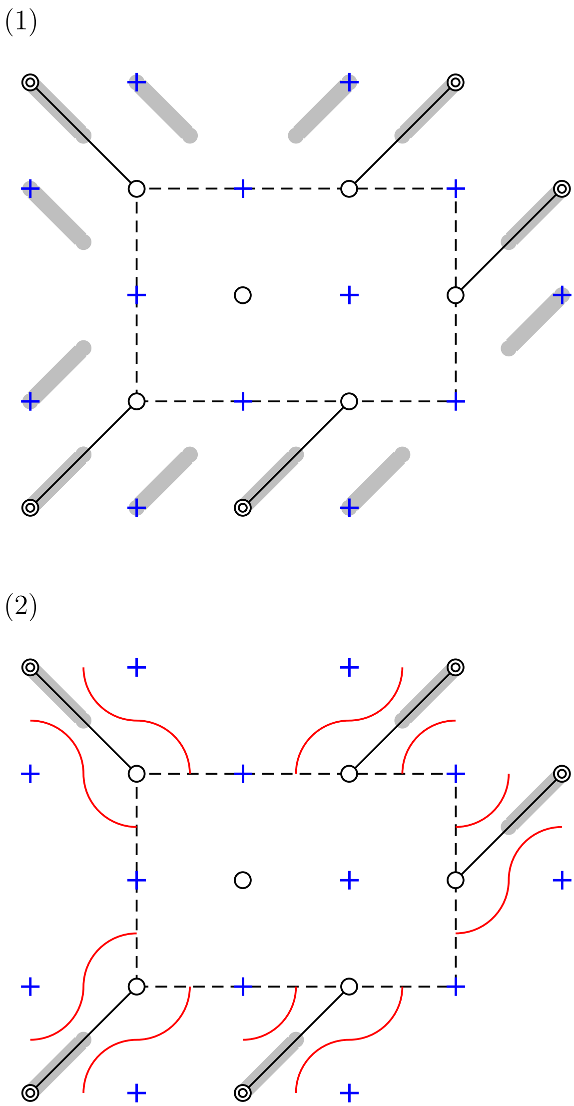



**Fig. 38** (**1**) We specify the dimers covering the vertices $V(G') \setminus V(G)$, so that $M \in \mathcal{D}(G)$ can be regarded as $M' \in \mathcal{D}(G')$. (**2**) The *slit curve* shows $\mathcal{D}_B(G)$ and $\mathcal{D}_B(G')$ are in one to one correspondence

random walk and the matrix tree theorem, we obtained an explicit formula for $|\mathcal{D}(G^{(1)})|$ and the probability of finding the impurity at given edge. These argument can not directly be applied to $k$ impurity case ($G^{(k)}$), because the number of configurations of impurity is not simply determined by the length of the tree, and the map $T_{\mathrm{DF}}$ in Theorem 2.8 (from the set $\mathcal{D}(G)$ of dimer coverings to the set of spanning trees on $\overline{G}_1$) is not surjective. Thus we can only deduce bounds for them.

We first introduce some notations. Let $G = G^{(k)}$, $N := |V(G)|$ and

$$\mathcal{T}(\overline{G}_1) := \{\text{spanning trees of } \overline{G}_1\},$$

$$\mathcal{T}(\overline{G}_1, Q) := \left\{T \in \mathcal{T}(\overline{G}_1) \mid \exists \left(T, S, \{e_j\}_{j=1}^{k}\right) \in \mathcal{F}(G, Q)\right\}$$

be the set of spanning trees of $\overline{G_1}$ and the set of those which corresponds to the dimer coverings of $G$ by the bijection given in Theorem 2.8. Moreover let $\overline{A} = \{\overline{a}_{ij}\}_{i,j=1,\dots,|G_1|+1}$ be a $(\mathbf{R}^{|G_1|+1} \times \mathbf{R}^{|G_1|+1})$-matrix given by

$$\overline{a}_{ij} := \begin{cases} \deg(i) & (i = j), \\ -1 & (\exists e = (i, j) \in E(\overline{G_1})), \\ 0 & (\text{otherwise}). \end{cases}$$

Let $A = \{a_{ij}\}_{i,j=1,\dots,|G_1|}$ be the restriction of $\overline{A}$ to $G_1$ and let $\mathbf{b} = (b_j)_{j=1}^{|G_1|}$, $\mathbf{p} = (p_j)_{j=1}^{|G_1|} \in \mathbf{R}^{|G_1|}$ be

$$b_j := \begin{cases} 1 & (j \text{ corresponds to a terminal}), \\ 0 & (\text{otherwise}), \end{cases}$$

$$\mathbf{p} := A^{-1}\mathbf{b}.$$

$p_j$ is equal to the probability that the random walk starting at $j$ hits the root $R$ through a terminal. For $j \in V_1$, let

$$E_2(j) := \big\{e = (x, j) \in E_2 \,|\, x \in V_2\big\}$$

be the set of edges in $E_2$ with $j$ being one of endpoint.

**Theorem 4.1** (1)

$$2^k|\mathcal{T}(\overline{G_1}, Q)| \le |\mathcal{D}(G)| \le |\det A| \left(\frac{4(N+k)}{k}\right)^k.$$

(2) *For $j \in V_1$, the probability of having an impurity on $j$ is estimated by*

$$\mathbf{P}(\exists \textit{ impurity} \in E_2(j)) \le C_k \left(\frac{2(N+k)}{k}\right)^k p_j \tag{4.1}$$

*where $C_k$ depends only on $k$.*

*Remark 4.2* $p_j$ generically decreases as $j$ is away from the terminals. In fact, it decays exponentially when $G_1$ is a one-dimensional chain (Sect. A.6), and decays polynomially when $G_1$ is a rectangle, as $j$ is away from the terminals, In general, the Combes-Thomas estimate gives us a bound for $p_j$ (e.g., [20])

$$p_j \le \frac{C}{d} \exp\Big(-cd \min_k\big(j, \{T_k\}\big)\Big)$$

where $d = d(0, \sigma(A))$, $\sigma(A)$ is the set of eigenvalues of $A$, and $C, c$ are some positive constants. Thus we have a reasonable answer to Conjecture 1.2 if the graph is one-dimensional or small enough. However, for large graphs better estimate is desirable to understand Conjecture 1.2 quantitatively, since the constant in RHS of (4.1) behaves like $N^k$ for large $N$.

*Remark 4.3* The random walk argument in the proof of Theorem 4.1 implies that, for the graph $G^{(k)}$, impurities are attracted to the terminals and this explains Conjecture 1.2 at least

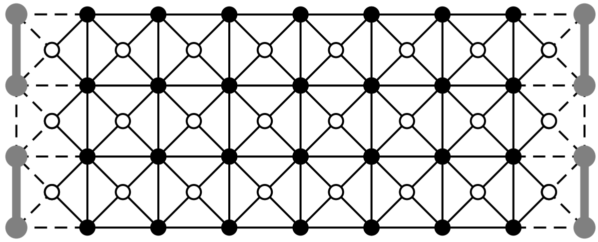


**Fig. 39** The Aztec rectangle with extra edges of size $7 \times 3$ can be regarded as a special case of $G^{(8,3)}$ where some impurities are fixed at the vertical part of boundaries

heuristically. For instance, in Fig. 8, terminals are put on the oblique side, to which impurities are attracted. For the graph $G^{(m,n)}$, we can embed $G^{(m,n)}$ into $G^{(k)}$ ($k = \frac{m+n+1}{2}$) as in the proof of Theorem 3.1. That is, in a sense, terminals are "hidden" on the boundary to which impurities are attracted. The Aztec rectangle can be regarded as a special case of $G^{(m,n)}$, where some of these impurities are fixed at the vertical part of boundaries (Fig. 39). Thus impurities are attracted to the horizontal part of boundaries (Fig. 6).

*Proof* (1) Let $f_j(T)$ be the number of impurity configurations of the $j$-th TI-tree composing $T \in \mathcal{T}(\overline{G_1}, Q)$. Then by Theorem 2.8,

$$F(T) := f_1(T) \cdot f_2(T) \cdot \cdots \cdot f_k(T) = \prod_{j=1}^{k} f_j(T)$$

is equal to the number of dimer coverings of $G$ corresponding to $T \in \mathcal{T}(\overline{G_1}, Q)$ hence

$$|\mathcal{D}(G)| = \sum_{T \in \mathcal{T}(\overline{G_1}, Q)} F(T). \tag{4.2}$$

Since $2 \le f_j(T) \le 4$ (length of the $j$-th tree) + 2, we have $2^k \le F(T) \le (\frac{4(N+k)}{k})^k$. Substituting it to (4.2) and using the matrix tree theorem yields Theorem 4.1(1).

(2) If we have an impurity on $E_2(j)$, then we find a TI-tree or a $T^l I$-tree containing the vertex $j$ through which $j$ is connected to the root. Let $C$ be the event given by

$$C := \{j \text{ is connected to the root through a terminal}\}.$$

Given $T \in \mathcal{T}(\overline{G_1}, Q)$, the number of configurations $F'(T)$ of the rest $(k-1)$ impurities is bounded from above by $F'(T) \le (\frac{4(N+k)}{k})^k$. Thus

$$\begin{aligned}
\mathbf{P}(\exists \text{ impurity} \in E_2(j)) &\le \frac{|\mathcal{T}(\overline{G_1})|}{|\mathcal{D}(G)|} \sum_{T \in \mathcal{T}(\overline{G_1}, Q)} F'(T) \frac{1_C}{|\mathcal{T}(\overline{G_1})|} \\
&\le \frac{|\mathcal{T}(\overline{G_1})| (\frac{4(N+k)}{k})^k}{|\mathcal{D}(G)|} \sum_{T \in \mathcal{T}(\overline{G_1})} \frac{1_C}{|\mathcal{T}(\overline{G_1})|} \\
&= \frac{|\mathcal{T}(\overline{G_1})| (\frac{4(N+k)}{k})^k}{|\mathcal{D}(G)|} p_j .
\end{aligned}$$

$1_C$ is the indicator function of the event $C$. It remains to substitute the lower bound for $|\mathcal{D}(G)|$ in Theorem 4.1(1) and use Lemma 4.4 below. □

**Lemma 4.4** *There are constants $C_k$ depending only on $k$ such that*

$$|\mathcal{T}(\overline{G_1})| \leq C_k |\mathcal{T}(\overline{G_1}, Q)|.$$

*Proof* We decompose $\mathcal{T}(\overline{G_1}) \setminus \mathcal{T}(\overline{G_1}, Q)$ into disjoint subsets as $\mathcal{T}(\overline{G_1}) \setminus \mathcal{T}(\overline{G_1}, Q) = \bigcup_{j=1}^{K} \mathcal{T}_j$ in terms of the kind of trees on each terminal, where $K$ depends only on $k$. Each $T \in \mathcal{T}_j$ can be changed into one in $\mathcal{T}(\overline{G_1}, Q)$ by a single flip in $T$. It then suffices to construct injections from $\mathcal{T}_j$ to $\mathcal{T}(\overline{G_1}, Q)$ defined by these flips. □

## Appendix A

In Appendices A.1–A.5, we discuss some implications of Theorems 2.1 and 2.8. In Appendix A.6, we apply Theorem 4.1 to the 1-dimensional chain.

### A.1 One Dimensional Strip

We consider $G = G^{(2k,1)}$ which is the strip of width 1 (Fig. 40). In this subsection we compute $D(2k) := |\mathcal{D}(G^{(2k,1)})|$ by using Theorem 2.1.

$D(2) = 8$ can be seen explicitly. By Theorem 2.1, the length of tree, composing the spanning forest satisfying the pairing condition, must be less than $2\sqrt{2}$, and if it lies on the end, it must be less than $\sqrt{2}$ (Fig. 41).

Hence if we put the block $G^{(2,1)}$ composed of two unit cubes to the left end of $G^{(2k,1)}$, the tree configuration must be one of the two shown in Fig. 42.

Therefore, taking possible configuration of impurities into consideration, we have $D(2k+2) = 2 \cdot 2 \cdot \frac{3}{2} \cdot D(2k) = 6D(2k)$ and thus

$$D(2k) = 8 \cdot 6^{k-1}.$$

It is also possible to deduce the same result by the transfer-matrix approach [15].

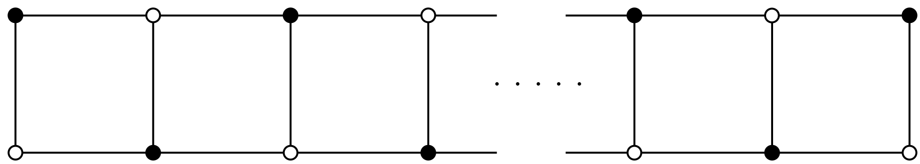


**Fig. 40** $G^{(2k,1)}$

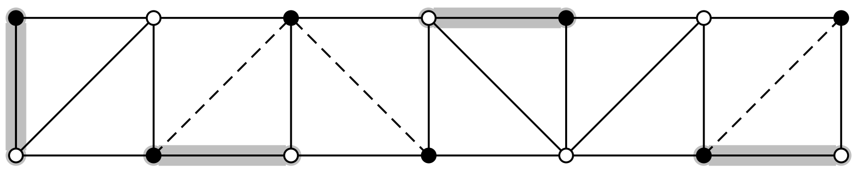


**Fig. 41** The length of tree must be less than $2\sqrt{2}$ (*inside*) and $\sqrt{2}$ (*end*)

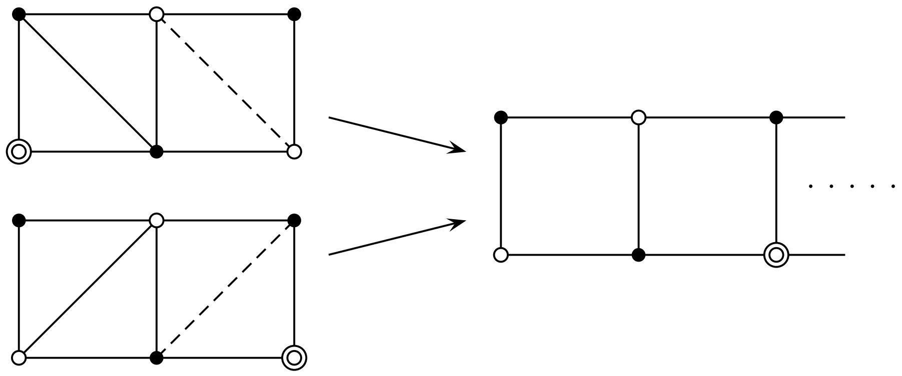


**Fig. 42** The tree configuration of $G^{(2k+2,1)}$ is given by putting one of the two $G^{(2,1)}$'s to the left end of $G^{(2k,1)}$

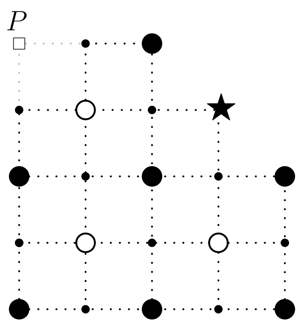



**Fig. 43** An example of $G$. The star corresponds to the root $R$

## A.2 Naive Description of Conjecture 1.2

The discussion in the proof of Theorem 2.1 gives us a naive explanation of Conjecture 1.2. By Proposition 2.3(0), impurities lying inside $G$ can always be moved to the boundary by successive applications of t-moves. Hence there are as many configurations with boundary impurities as those with inner impurities. On the other hand, there are some configurations in which most curves lie near the boundaries and do not enter inside. Therefore the number of configuration with boundary impurities would be much more than that with inner impurities. Furthermore the results of simulations seems to imply that almost all slit curves typically lie near the boundary whereas there are only few huge ones inside $G$.

## A.3 Relation to the Temperley Bijection

In our notation, the Temperley bijection is stated as follows. Consider $G^{(1)}$ as in Sect. 2.2, eliminate a vertex $P \in V_2$ and these edges such that $P$ is one of their endpoint, and set $G := G^{(1)} \setminus \{P\}$. Figure 43 gives an example.
Temperley bijection gives a bijection between the following two sets.

$$\mathcal{D}(G) := \{\text{dimer coverings of } G\},$$

$$\mathcal{T}(\overline{G_1}) := \{\text{spanning trees of } \overline{G_1}\}.$$

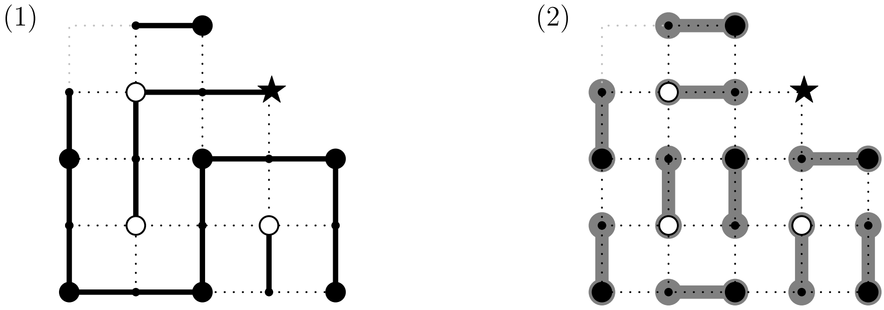



**Fig. 44** (**1**) An example of spanning tree of $\overline{G_1}$, (**2**) the corresponding dimer covering of $G$

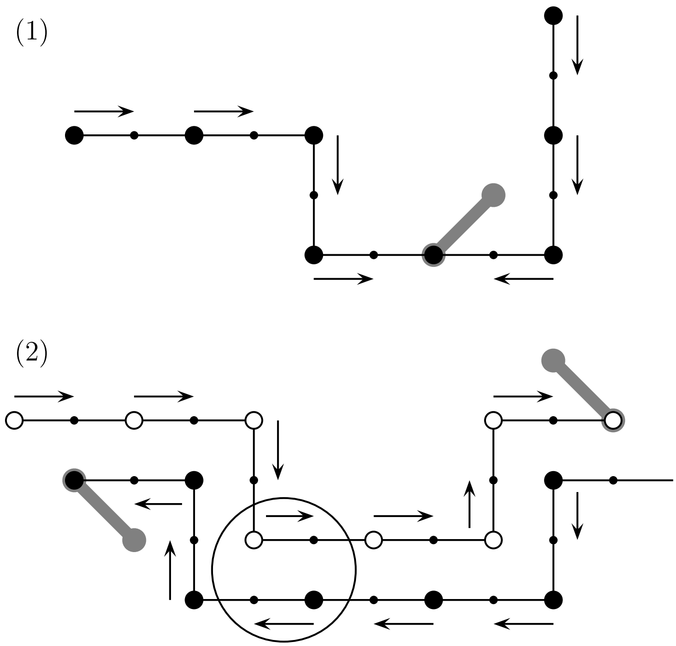



**Fig. 45** (**1**) Dimers are arranged along the orientation of trees, (**2**) if a pair of neighboring dimers have the opposite orientation, s-move is possible there

This bijection is similar to that in Corollary 2.11 where the impurity plays a role of the vertex $P$. Figure 44(1), (2) describes an example of the Temperley bijection which corresponds to the dimer covering described in Fig. 33.

### A.4 Local-move Connectedness

In the proof of Proposition 2.4, we constructed the dimer covering from a triple $(F_1, F_2, \{e_j\}) \in \mathcal{F}(G, P)$. On the trees of $F_1$, $F_2$, we can introduce orientation by regarding the impurities as the roots of trees. We then note the following facts. (1) Dimers on those trees are arranged along this orientation (Fig. 45(1)). (2) Let $T_j$ $(j = 1, 2)$ be trees of the

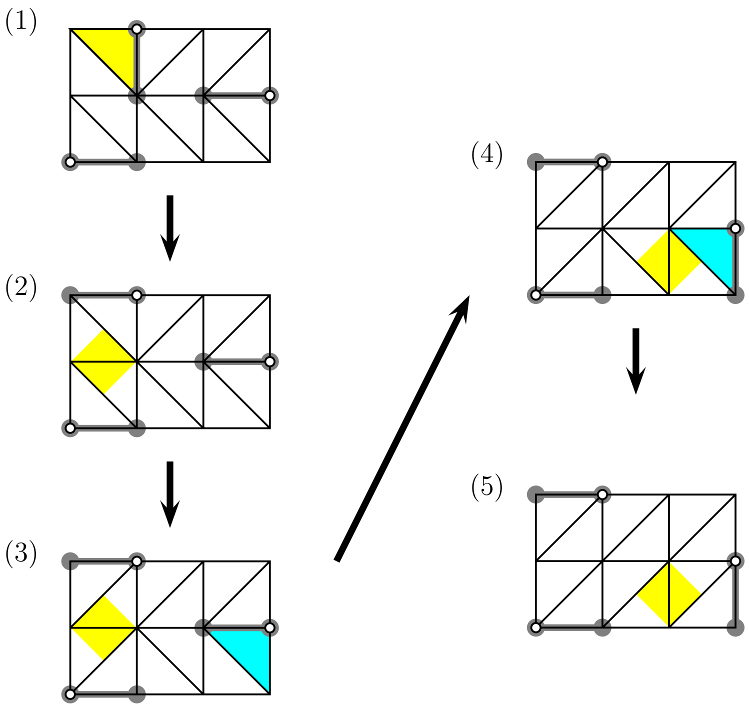



**Fig. 46** Any dimer covering (**1**) is transformed to the specific one (**5**)

spanning forest of $G_j$ and let $e_j \in T_j$ be some neighboring dimers which are parallel each other. If $T_1$ and $T_2$ do not share the same impurity, and if $e_1, e_2$ have the opposite orientation, then s-move is possible at $e_1, e_2$ (Fig. 45(2)).

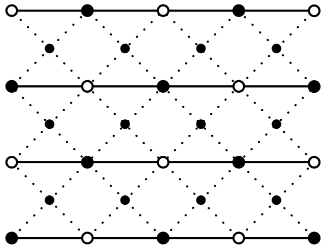
By moving impurities by t-moves, we can adjust the orientation of each trees so that s-move is possible at given site with a pair of dimers belonging to different trees. In fact, we have an (not so simple) algorithm by which any dimer covering can be transformed to the specific one where all trees are parallel (Fig. 46). These facts give us another proof of LMC.

### A.5 Application to Other Graphs

The argument of Theorems 2.1, 2.8 may partially be applied to the Bow-tie lattice, the triangular lattice, and the hexagonal lattice, which we briefly discuss below.

(1) Bow-tie lattice
The Bow-tie lattice $G_B$ is obtained by removing vertical edges which connect vertices in $V_1$, $V_2$ from those in $G = G^{(m,n)}$ (Fig. 47). Theorem 2.1 can be directly applied, except that impurities in the vertical direction are not allowed. Since $V(G_B) = V(G)$ and $E(G_B) \subset E(G)$, the set $\mathcal{D}(G_B)$ of the dimer coverings of $G_B$ satisfies $\mathcal{D}(G_B) \subset \mathcal{D}(G)$, from which we deduce the following facts. (a) We can define a pair of local moves under which we have LMC. (b) If we specify the location of impurities, then the number of dimer coverings on $G$ and $G_B$ are equal. Hence a proof of Conjecture 1.2 for $G$ would also prove that for $G_B$.

**Fig. 47** Bow-tie lattice



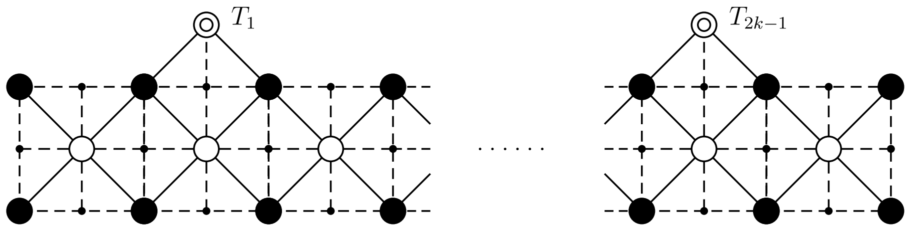



**Fig. 48** 1-dimensional chain with $(2k-1)$-terminals

(2) Triangular lattice
By Theorem 2.1, the dimer covering on the triangular lattice $G_T$ is reduced to that of the Bow-tie lattice with holes.

(3) Hexagonal lattice
The hexagonal lattice with extra edges is also introduced in [17], and one can prove a bijection theorem as Theorems 2.1, 2.8 for this graph. However, this bijection is not useful to study LMC and the enumeration of dimer coverings, while the method in [16] works well to prove LMC. The same phenomenon as in Fig. 5 is observed.

### A.6 1-dimensional Chain with $k$-impurities

In this subsection we briefly discuss $G^{(k)}$ where $G_1$ is the 1-dimensional chain with length $2L+1$ (Fig. 48) and show that Conjecture 1.2 is true in this case. We embed $G_1$ into $\mathbf{Z}$ and suppose that $V(G_1) = \{-L, -L+1, \ldots, L-1, L\}$. We fix $t_1, t_2, \ldots, t_{2k-1} \in V(G_1)$ and attach terminals $T_j$ to $t_j$ $(j = 1, 2, \ldots, 2k-1)$.
Let

$$H = H_0 + V,$$

be a discrete Schrödinger operator on $l^2(\mathbf{Z})$ where the free Laplacian $H_0$ and the potential $V$ are given respectively by

$$H_0(x,y) := \begin{cases} 4 & (x=y), \\ -1 & (|x-y|=1), \\ 0 & (\text{otherwise}), \end{cases} \qquad V(x) := \begin{cases} -\frac{1}{2} & (x = t_j,\ j = 1, 2, \ldots, 2k-1), \\ 0 & (\text{otherwise}). \end{cases}$$

Since $H$ is a finite-rank perturbation of $H_0$, we can find a finite set $F \subset \mathbf{R}$ such that the spectrum of $H$ satisfies (e.g., [1])

$$\sigma(H) = [2,6] \cup F.$$

**Theorem 1.1** *Assume $0 \notin F$. Then we can find positive constants $\rho > 0$, $C_k > 0$ depending only on $k$ such that*

$$\mathbf{P}(\exists \mathit{impurity} \in E_2(x)) \le C_k L^k \exp\Big(-\rho \min_{1\le j\le 2k-1} |x - t_j|\Big), \quad x \in V(G_1).$$

We can explicitly check the condition $0 \notin F$ by a transfer matrix calculation, and in fact this condition is generically true. If $L$ is large and $\min_{1\le j\le 2k-1} |x - t_j|$ is of order $L$, then the probability in question is exponentially small and thus Conjecture 1.2 is true in this case. For a proof of Theorem 1.1, let $H_L := H|_{V(G_1)}$ and let $A_L$ be the matrix associated to $G_1$ defined in Sect. 4. We then have $0 \in \sigma(A_L)$ if and only if $0 \in \sigma(H_L)$. By the condition $0 \notin F$ and a perturbative argument, we can find a $L$-independent constant $\delta > 0$ such that $d(0, \sigma(A_L)) \ge \delta$ which gives us an exponential decay estimate of the matrix elements of $A_L^{-1}$ (e.g., [20]). Together with Lemma 4.4, we have the desired bound.

**Acknowledgement** The authors would like to thank Yusuke Higuchi and Tomoyuki Shirai for valuable discussions and comments. They would also like to thank referees for their helpful comments and remarks which improved this manuscript.

## References

1. Albeverio, S., Gesztesy, F., Hoegh-Krohn, R., Holden, H.: Solvable Models in Quantum Mechanics. Springer, Berlin (1988)
2. Burton, R., Pemantle, R.: Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances. Ann. Probab. **21**, 1329–1371 (1993)
3. Ciucu, M.: A random tiling model for two dimensional electrostatics. Mem. Am. Math. Soc. **178**(839), 1–106 (2005)
4. Ciucu, M.: The scaling limit of the correlation of holes on the triangular lattice with periodic boundary conditions. Mem. Am. Math. Soc. **199**, 935 (2009) (English summary)
5. Cohn, H., Elkies, N., Propp, J.: Local statistics for random domino tilings of the Aztec diamond. Duke Math. J. **85**(1), 117–166 (1996)
6. Dhar, D.: Self-organized critical state of sandpile automaton models. Phys. Rev. Lett. **64**(14), 1613–1616 (1990)
7. Heilmann, O.J., Lieb, E.H.: Theory of monomer-dimer systems. Commun. Math. Phys. **25**, 190–232 (1972)
8. Kasteleyn, P.W.: The statistics of dimers on a lattice I. The number of dimer arrangements on a quadratic lattice. Physica **27**, 1209–1225 (1961)
9. Kasteleyn, P.W.: Dimer statistics and phase transitions. J. Math. Phys. **4**, 287–293 (1963)
10. Kenyon, R.: The planar dimer model with boundary: a survey. In: Baake, M., Moody, R. (eds.) Directions in Mathematical Quasicrystals. CRM Monograph Series, vol. 13, pp. 307–328. American Mathematical Society, Providence (2000)
11. Kenyon, R.W., Propp, J.G., Wilson, D.B.: Trees and matchings. Electron. J. Comb. **7**, 25–34 (2000) Research paper
12. Kenyon, C., Rémila, E.: Perfect matchings in the triangular lattice. Discrete Math. **152**, 192–210 (1996)
13. Luby, M., Randall, D., Sinclair, A.: Markov chain algorithms for planar lattice structures. SIAM J. Comput. **31**(1), 167–192 (2001)
14. Nakano, F., Sadahiro, T.: Domino tilings with diagonal impurities, arXiv:0901.4824
15. Nakano, F., Sadahiro, T.: A note on perfect matchings in fixed-width Aztec rectangles with extra edges, in preparation

16. Nakano, F., Ono, H., Sadahiro, T.: Connectedness of domino tilings with diagonal impurities. Discrete Math. doi:10.1016/j.disc.2010.02.015
17. Propp, J.: Enumeration of matchings: problems and progress. In: New Perspectives in Algebraic Combinatorics, Berkeley, CA, 1996–1997. Math. Sci. Res. Inst. Publ., vol. 38, pp. 255–291. Cambridge University Press, Cambridge (1999)
18. Temperley, H., Fisher, M.E.: The dimer problem in statistical mechanics—an exact result. Philos. Mag. **6**, 1061–1063 (1961)
19. Thurston, W.P.: Groups, tilings and finite state automata. In: Summer 1989 AMS Colloquium Lectures
20. Kirsch, W.: An invitation to random Schrödinger operators. In: Random Schrödinger Operators, Panoramas st Synthéses, vol. 25 (2008)